Resolution Folding under φ–π–e Triple-Operator Constraints: A Zeckendorf–Hilbert–Abel Framework for the $64 \to 21$ Projection and Recursive Uplift

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Abstract

We define a purely mathematical notion of *resolution folding*: a computable map from a finite microstate readout to a smaller stable type space selected by constraint channels. At minimal local resolution, the microstate space is $\Omega_6 := \{0,1\}^6$ with $|\Omega_6| = 64$ and linearization $\mathcal{H}_6 := \ell^2(\Omega_6)$.

The stability mechanism is organized into three commuting channels, denoted φ–π–e. The φ-channel is the golden-mean (Zeckendorf) grammar constraint forbidding adjacent ones. The π-channel is a cyclic-closure refinement inside the admissible sector, induced by wrap-around admissibility and periodic-orbit counting. The e-channel is an analytic stability viewpoint expressed by Artin–Mazur zeta functions and spectral normalization; we also record a weighted one-parameter extension in which e becomes genuinely distinct from φ without changing the forbidden-word grammar.

While the Fibonacci count of admissible words is classical, the finite-resolution folding is made explicit and quantitative. We define a computable surjection $\mathrm{Fold}_6 : \{0, \ldots, 63\} \twoheadrightarrow X_6$ by Zeckendorf normalization and window truncation, and we give closed-form preimage sets and an exact degeneracy histogram for Fold_6. Beyond the base case, we provide a general preimage characterization for Fold_m and derive quantitative growth bounds for folding degeneracy.

To connect folding to scale change, we couple Hilbert addressing (spatial refinement and its dihedral layout group D_4) with Zeckendorf window growth (syntactic refinement). Finally, we state two non-premise application interfaces—a codon-to-type compression template and a three-channel factorization template—formulated as falsifiable mapping problems rather than assumptions.

Keywords: Zeckendorf representation; golden mean shift; shifts of finite type; Artin–Mazur zeta function; Abel normalization; pole barrier; monodromy; Hilbert curve; dihedral group; resolution folding.

Conventions. Unless otherwise stated, log denotes the natural logarithm. We use $w = w_1 \cdots w_m$ for a finite binary word with letters $w_i \in \{0,1\}$, and reserve n for Hilbert resolution order and m for Zeckendorf window length. The standard Abel path refers to the limit process $r \uparrow 1$ with $r \in (0,1)$, applied to Abel generating functions defined (and holomorphic) for $|r| < 1$.

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Contents

1 Introduction

1.1 Problem statement: from 64 microstates to a stable 21-type sector

Fix a finite local readout space with 64 microstates,

$$\Omega_6 := \{0,1\}^6, \qquad |\Omega_6| = 2^6 = 64,$$

and its linearization $\mathcal{H}_6 := \ell^2(\Omega_6) \cong \mathbb{C}^{64}$. Our goal is to formalize, in a purely mathematical language, a notion of *resolution folding*: how a finite-resolution readout can admit a computable projection from the full microstate set to a smaller, stable effective set, and how this mechanism can iterate across scales.

Definition 1.1 (Resolution folding scheme (finite window))**.** *Fix a window length $m \geq 1$ and let $\Omega_m := \{0,1\}^m$ be the microstate alphabet. A* resolution folding scheme *at window length m consists of:*

- *a* stable type set $X_m \subseteq \Omega_m$ *selected by constraint channels (equivalently, $X_m = \{w \in \Omega_m : \mathcal{D}(w) = 0\}$ for some nonnegative defect function $\mathcal{D}$);*
- *a surjective* folding map $\mathsf{Fold}_m : \Omega_m \twoheadrightarrow X_m$ *(or, after indexing $\Omega_m \cong \{0, \dots, 2^m - 1\}$, a surjection from $\{0, \dots, 2^m - 1\}$ onto X_m);*
- *the induced orthogonal projection $P_m : \ell^2(\Omega_m) \to \ell^2(\Omega_m)$ onto $\ell^2(X_m)$.*

A scale interface *is a family of such schemes as m varies, together with natural maps relating (X_m, Fold_m) across different m (Section 7).*

Remark 1.2 (Relation to companion manuscripts)**.** *Motivations for coupling address families, canonical codings, and Abel-type regularizations appear in the broader HPA–Ω manuscript series; see, e.g., [1–3]. The present paper is self-contained and uses only the abstract mathematical structures needed for the $64 \to 21$ folding analysis.*

Related work and positioning. The golden mean constraint defines a shift of finite type and is among the standard model examples in symbolic dynamics; see [4]. The determinant/zeta identities used here are classical for topological Markov shifts and sit within thermodynamic formalism and transfer-operator methods; see [5–7] for general frameworks and refinements (pressure, weighted zeta functions, and Ruelle operators). Zeckendorf and Fibonacci numeration, and associated algorithms, are standard in the combinatorics on words and automatic sequences literature; see [8, 9]. The Hilbert curve and its use as a locality-preserving ordering (e.g. in indexing and embedding contexts) are classical; see [10–12].

1.2 Contributions (and what is classical)

At fixed word length 6, the golden-mean forbidden grammar (*no adjacent ones*) selects precisely 21 admissible binary words. This Fibonacci count and its asymptotic growth rate are classical. The goal of this paper is not to re-prove standard symbolic-dynamics background, but to make the *folding mechanism* explicit and quantitative at a fixed finite resolution, while keeping a clean interface to higher-resolution generalizations.

Concretely, beyond standard background, the paper contributes:

- **An explicit folding map with closed preimages.** We define a computable surjection $\text{Fold}_6 : \{0, \dots, 63\} \twoheadrightarrow X_6$ via Zeckendorf normalization and window truncation, and we give closed-form preimage sets and an exact degeneracy histogram (Theorem 6.15; Appendix E).

- **Quantitative control beyond the base case.** We provide a general preimage characterization for Fold_m and derive explicit degeneracy bounds and scaling laws for the $2^m \to F_{m+2}$ compression (Section 6).
- **A symmetry-invariant spatial interface.** We formalize the dihedral layout family D_4 for Hilbert addressing and prove exact invariance statements for adjacency-based statistics under layout changes (Section 2).
- **An analytic stability channel with a separable extension.** We formulate the e-channel via zeta/Abel normalization and record a weighted finite-state extension in which analytic stability is not equivalent to a single forbidden word (Section 5).

With these pieces in place, we enrich the φ-stable sector with two additional, commuting channels:

- **(π) discrete monodromy / closure.** Within the admissible set, a cyclic wrap-around condition splits the 21 stable words into 18 cyclically admissible states and 3 boundary states.
- **(e) analytic stability / Abel–zeta pole barrier.** The golden-mean shift has zeta function $\zeta(z) = 1/(1 - z - z^2)$; after spectral normalization $z = r/\varphi$, the normalized zeta is holomorphic on $|r| < 1$ with its principal pole on the boundary.

1.3 Main results (finite resolution)

Let $X_6 \subset \Omega_6$ be the set of length-6 words with no occurrence of the substring 11. Our main finite-resolution statements are:

- **Stable dimension ($64 \to 21$).** The orthogonal projection $P_\varphi : \mathcal{H}_6 \to \mathcal{H}_6$ onto $\ell^2(X_6)$ has rank $|X_6| = 21$.
- **Canonical split ($21 = 18 \oplus 3$).** The cyclic admissibility condition (wrap-around allowed transition) defines a partition

  $$X_6 = X_6^{\mathrm{cyc}} \sqcup X_6^{\mathrm{bdry}}, \qquad |X_6^{\mathrm{cyc}}| = 18, \quad |X_6^{\mathrm{bdry}}| = 3,$$

  with $X_6^{\mathrm{bdry}} = \{100001, 100101, 101001\}$.
- **A computable folding map.** We define a natural surjection

  $$\mathrm{Fold}_6 : \{0, \ldots, 63\} \twoheadrightarrow X_6$$

  by taking the Zeckendorf representation of an integer and truncating to a length-6 window.
- **Strong degeneracy law.** The full preimage structure of Fold_6 on $\{0, \ldots, 63\}$ is explicit: every output has degeneracy 2, 3, or 4, with histogram $(8, 4, 9)$ for $(2, 3, 4)$, and closed-form preimage sets (Theorem 6.15).

1.4 Quantitative summary at a glance

Table 1 records the core finite-resolution quantities and their general formulas (when available).

Quantity	General formula	At window length 6
Microstate count	$\lvert\Omega_m\rvert = 2^m$	$\lvert\Omega_6\rvert = 64$
Admissible (stable) types	$\lvert X_m\rvert = F_{m+2}$	$\lvert X_6\rvert = 21$
Cyclically admissible types	$\lvert X_m^{\mathrm{cyc}}\rvert = \mathrm{Tr}(A^m) = F_{m-1} + F_{m+1}$	$\lvert X_6^{\mathrm{cyc}}\rvert = 18$
Boundary types	$\lvert X_m^{\mathrm{bdry}}\rvert = F_{m-2}$ (for $m \geq 2$)	$\lvert X_6^{\mathrm{bdry}}\rvert = 3$
Golden mean zeta	$\zeta(z) = 1/\det(I - zA)$	$\zeta(z) = 1/(1 - z - z^2)$
Spectral normalization	$z = r/\rho(A)$, pole at $r = 1$	$\rho(A) = \varphi$, pole at $r = 1$
Mean folding degeneracy	$\frac{2^m}{\lvert X_m\rvert} = \frac{2^m}{F_{m+2}}$	$\frac{64}{21}$
Fold_6 degeneracy histogram	explicit finite classification	$(2 \mapsto 8,\ 3 \mapsto 4,\ 4 \mapsto 9)$

Table 1: Quantitative summary of the $64 \to 21$ folding model and its generalizations.

1.5 Resolution as recursion: Hilbert addressing and Zeckendorf windowing

Two recursive structures provide the scale interface. On the spatial side, finite-resolution Hilbert addressing maps a one-dimensional tick index to a two-dimensional $2^n \times 2^n$ grid, with an 8-element layout group D_4 acting by global symmetries. On the syntactic side, increasing the Zeckendorf window length m grows the admissible set size as $|X_m| = F_{m+2}$ (Fibonacci recursion), with asymptotic rate $|X_m| \asymp \varphi^m$. We view resolution folding as a coupling of these two recursions: spatial refinement changes the embedding/locality model, while syntactic refinement changes the stable type space.

1.6 Roadmap

Section 2 defines the 64-state model and Hilbert address families. Section 3 develops the φ-constraint and proves the 21 count. Section 4 introduces the π-channel and the $18 \oplus 3$ split. Section 5 develops the Abel–zeta pole-barrier viewpoint and formalizes the e-channel at finite window. Section 6 defines the folding map Fold_6 and proves surjectivity. Section 7 discusses recursion across Hilbert order and Zeckendorf window. Section 8 packages the construction as commuting defect operators and states a conservative iterability axiom. Section 9 records two application interfaces as falsifiable mapping problems.

2 Discrete state space and address families

2.1 The 64-state local readout space

We work at fixed local readout length 6. Define

$$\Omega_6 := \{0,1\}^6, \qquad \mathcal{H}_6 := \ell^2(\Omega_6) \cong \mathbb{C}^{64},$$

with the standard orthonormal basis $\{\mathbf{e}_w : w \in \Omega_6\}$. Elements $w \in \Omega_6$ will be referred to as *microstate labels.*

Binary indexing convention. We will also use the canonical identification between Ω_6 and integer indices $\{0, 1, \ldots, 63\}$ given by binary expansion. Define

$$\mathrm{int}_6 : \Omega_6 \to \{0, 1, \ldots, 63\}, \qquad \mathrm{int}_6(w_1 \cdots w_6) := \sum_{j=1}^{6} w_j\, 2^{6-j}.$$

Its inverse $\mathrm{bin}_6 : \{0, \ldots, 63\} \to \Omega_6$ maps an integer to its 6-bit binary word. This convention turns statements about 64 indices into equivalent statements about 6-bit words; alternative identifications correspond to a change of readout basis.

2.2 Hilbert addressing at finite spatial resolution

Let $n \geq 1$ and consider the square lattice

$$\Lambda_n := \{0, 1, \ldots, 2^n - 1\}^2, \qquad |\Lambda_n| = 2^{2n}.$$

A finite-resolution Hilbert address map is a bijection

$$H_n : \{0, 1, \ldots, 2^{2n} - 1\} \to \Lambda_n$$

with a one-step locality property: consecutive indices map to nearest neighbors on the grid, i.e.

$$\|H_n(t+1) - H_n(t)\|_1 = 1 \qquad (0 \leq t < 2^{2n} - 1),$$

where $\|\cdot\|_1$ is the ℓ^1 (Manhattan) norm on $\mathbb{Z}^2$. Such maps are standard discrete versions of the Hilbert space-filling curve; see, e.g., [10, 11]. They are also widely used as locality-preserving orderings in applications (e.g. clustering behavior for range queries); see [12].

For the purposes of this paper, the relevant point is the cardinality match: at order $n = 3$ one has $|\Lambda_3| = 2^6 = 64$, matching the microstate count. Thus a resolution-3 screen lattice provides a canonical geometric carrier for a 64-state slice.

Remark 2.1 (Scan indices as words)**.** *At order $n = 3$, a scan index $t \in \{0, \ldots, 63\}$ can be viewed as a word $\mathrm{bin}_6(t) \in \Omega_6$. Therefore any constraint or folding map defined on Ω_6 can be pulled back to scan indices (and thus to screen sites via H_3).*

2.3 Layout group and 8 address families

Let $G := D_4$ be the dihedral group of the square (four rotations and four reflections), with $|G| = 8$. Each element $g \in G$ acts on Λ_n by an isometry, hence produces a new address map

$$H_n^{(g)} := g \circ H_n.$$

We interpret $\{H_n^{(g)} : g \in G\}$ as an *address family* of size 8 at fixed resolution. Changing g preserves bijectivity and locality but changes how local neighborhoods are compiled into a scan order.

Proposition 2.2 (D4 invariance of adjacency-based scan-order statistics)**.** *Fix a resolution n and a bijective address map H_n. Let $g \in D_4$ and define $H_n^{(g)} = g \circ H_n$. Let $\sim$ denote nearest-neighbor adjacency on Λ_n. Then for any function $f : \{0, \ldots, 2^{2n} - 1\} \to \mathbb{R}$ and any edge functional of the form*

$$\mathcal{S}(H, f) := \sum_{\substack{x \sim y \\ x, y \in \Lambda_n}} \Phi\big(f(H^{-1}(x)),\, f(H^{-1}(y))\big),$$

where $\Phi : \mathbb{R}^2 \to \mathbb{R}$ is any fixed function, one has

$$\mathcal{S}(H_n^{(g)}, f) = \mathcal{S}(H_n, f).$$

Proof. Since g is a graph automorphism of $(\Lambda_n, \sim)$, the map $(x, y) \mapsto (g^{-1}x, g^{-1}y)$ is a bijection of the edge set. Moreover, $(H_n^{(g)})^{-1}(x) = H_n^{-1}(g^{-1}x)$. Substituting and changing variables over edges yields equality. □

Remark 2.3 (Concrete invariants)**.** *Examples covered by Proposition 2.2 include:*

- *the histogram of scan-index separations $|H^{-1}(x) - H^{-1}(y)|$ over neighboring pairs $x \sim y$;*
- *label-correlation counts for label fields of the form $x \mapsto \mathrm{Fold}_m(H^{-1}(x))$;*

- *label-correlation counts for label fields of the form* $x \mapsto \mathsf{Fold}_m^{\mathrm{bin}}(\mathrm{bin}_m(H^{-1}(x)))$*;*

- *average Hamming distance between labels across adjacent sites (with* Φ *chosen accordingly).*

Thus the dihedral layout family changes the embedding only by a symmetry, and adjacency-based statistics are exactly layout-invariant.

Corollary 2.4 (Geometric rigidity under the layout group)**.** *For fixed* n*, the set of layouts* $\{H_n^{(g)} : g \in D_4\}$ *is a* D_4*-orbit under left composition, and any two layouts differ by a global lattice isometry. Consequently, within this model class, any statement about a pulled-back label field* $x \mapsto \ell(H_n^{-1}(x))$ *that depends only on the adjacency relation* $x \sim y$ *is independent of the chosen layout.*

Proof. For $g, h \in D_4$ one has $h \circ H_n^{(g)} = (hg) \circ H_n$, so the family is a D_4-orbit. If $H_n^{(g_1)}$ and $H_n^{(g_2)}$ are two layouts, then $H_n^{(g_2)} = (g_2 g_1^{-1}) \circ H_n^{(g_1)}$ with $g_2 g_1^{-1} \in D_4$ a lattice isometry. The final assertion follows from Proposition 2.2. □

2.4 Two-sector layout space $G \times G$ and a 6-bit encoding

In many hierarchical constructions one considers two independent address families (two sectors) on the same lattice. Abstractly this means choosing $(g_+, g_-) \in G \times G$, hence defining a layout state space

$$\Omega_{\mathrm{layout}} := G \times G, \qquad |\Omega_{\mathrm{layout}}| = 8 \cdot 8 = 64.$$

Therefore there exists a (non-canonical) bijection

$$\iota : \Omega_{\mathrm{layout}} \xrightarrow{\sim} \Omega_6.$$

We do *not* fix ι in the main text; different choices correspond to different readout bases and encodings. Appendix B records a concrete 6-bit encoding obtained from the normal form $g = r^k s^\epsilon$ in D_4.

Remark 2.5 (Why Hilbert addressing appears here)**.** *The* φ–π–e *constraints are formulated at the level of words* $w \in \Omega_6$*. Hilbert addressing enters as the* resolution mechanism*: it specifies how a one-dimensional scan index is compiled into a spatial neighborhood structure on a screen lattice at a given resolution, and it provides a natural symmetry group (the* D_4 *layout group). This yields a clean separation: constraints define stable types; addressing defines how those types are embedded and refined across spatial scales.*

3 The phi-constraint: Zeckendorf legality and the golden mean subshift

3.1 Golden mean grammar at finite window length

Let $\mathcal{F} := \{11\}$ be the forbidden word set. Define the admissible set of length-6 words

$$X_6 := \{w \in \{0,1\}^6 : \ w \text{ contains no occurrence of } 11\}.$$

Equivalently, $w = w_1 \cdots w_6 \in X_6$ if and only if $w_i w_{i+1} = 0$ for $i = 1, \ldots, 5$. This is the length-6 truncation of the classical *golden mean shift* (a shift of finite type). It also coincides with the locality rule for Zeckendorf representations in the golden Ostrowski degeneration; see, e.g., [8, 9, 13].

3.2 Counting admissible words: $|X_6| = 21$

Let a_n denote the number of binary words of length n with no adjacent ones.

Proposition 3.1 (Fibonacci recursion for no-adjacent-one words)**.** *For $n \geq 3$ one has*

$$a_n = a_{n-1} + a_{n-2}, \qquad a_1 = 2, \quad a_2 = 3,$$

hence $a_n = F_{n+2}$ where (F_k) is the Fibonacci sequence with $F_1 = 1, F_2 = 1$.

Proof. Classify admissible words by their first letter. If the first letter is 0, the remaining $n-1$ letters form any admissible word, contributing a_{n-1} possibilities. If the first letter is 1, the second letter must be 0, and the remaining $n-2$ letters form any admissible word, contributing a_{n-2}. Thus $a_n = a_{n-1} + a_{n-2}$. The initial values are $a_1 = 2$ (words $0, 1$) and $a_2 = 3$ (words $00, 01, 10$). □

Corollary 3.2 (The $64 \to 21$ count)**.** *One has $|X_6| = a_6 = F_8 = 21$.*

3.3 The phi-stable subspace as an orthogonal projection

Define the orthogonal projector

$$P_\varphi : \mathcal{H}_6 \to \mathcal{H}_6, \qquad P_\varphi(\mathbf{e}_w) = \begin{cases} \mathbf{e}_w, & w \in X_6, \\ 0, & w \notin X_6. \end{cases}$$

Proposition 3.3 (φ-stability is 21-dimensional)**.** *The operator P_φ is an orthogonal projection and*

$$\operatorname{rank}(P_\varphi) = \dim \operatorname{Im}(P_\varphi) = |X_6| = 21.$$

Proof. By construction P_φ is diagonal in the orthonormal basis $\{\mathbf{e}_w\}$ with eigenvalues 0 or 1. Hence $P_\varphi^2 = P_\varphi = P_\varphi^*$ and its image is $\ell^2(X_6)$, of dimension $|X_6| = 21$. □

Remark 3.4 (Entropy rate and the appearance of φ)**.** *The appearance of φ is not only combinatorial but also dynamical. The golden mean shift is a shift of finite type with adjacency matrix $A = \left(\begin{smallmatrix} 1 & 1 \\ 1 & 0 \end{smallmatrix}\right)$, hence its topological entropy equals $\log \rho(A) = \log \varphi$; see [4]. At the finite-window level, Proposition 3.1 gives $|X_n| = F_{n+2}$, so $\frac{1}{n} \log |X_n| \to \log \varphi$ as $n \to \infty$ by the standard Fibonacci asymptotics. In this sense φ is the canonical growth/grammar constant for the stability constraint "no adjacent ones".*

Proposition 3.5 (Entropy from admissible growth)**.** *Let $X_n \subset \{0,1\}^n$ be the length-n golden mean language (no adjacent ones). Then*

$$\lim_{n \to \infty} \frac{1}{n} \log |X_n| = \log \varphi.$$

Proof. Proposition 3.1 gives $|X_n| = F_{n+2}$. By Binet's formula (or any standard Fibonacci bound), $F_{n+2} = \frac{\varphi^{n+2} - (-\varphi)^{-n-2}}{\sqrt{5}}$, hence $F_{n+2} = \Theta(\varphi^n)$. Taking logs and dividing by n yields the limit $\log \varphi$. □

4 The pi-channel: discrete monodromy and cyclic closure

In continuous settings, "monodromy" and "single-valuedness" are expressed via loops and holonomy. Here we use the word *monodromy* only as a mnemonic for a discrete *closure* test: a finite word determines a directed path on a constraint graph, and closure is detected by a wrap-around admissibility condition. For clarity, the mathematical object of interest in this paper is *cyclic closure* (periodic-orbit compatibility), not a continuous holonomy.

4.1 The golden mean graph

The golden mean shift admits a two-state Markov presentation. Let the vertex set be $V := \{0, 1\}$ and define the allowed directed edges

$$E := \{0 \to 0,\ 0 \to 1,\ 1 \to 0\},$$

excluding $1 \to 1$. The adjacency matrix is

$$A = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}.$$

Every word $w = w_1 \cdots w_6 \in X_6$ determines a directed path

$$w_1 \to w_2 \to \cdots \to w_6$$

of length 5 on this graph.

4.2 Endpoint defect via a boundary operator

Let $\mathbb{Z}V$ be the free abelian group generated by V, and $\mathbb{Z}E$ the free abelian group generated by E. Define the boundary operator $\partial : \mathbb{Z}E \to \mathbb{Z}V$ by

$$\partial(u \to v) := v - u.$$

For $w \in X_6$, let $c(w) \in \mathbb{Z}E$ be the path chain obtained by summing the edges traversed by the path $w_1 \to \cdots \to w_6$. Then

$$\partial c(w) = w_6 - w_1 \in \mathbb{Z}V.$$

We call w *endpoint-closed* if $\partial c(w) = 0$, i.e. $w_1 = w_6$, and *endpoint-open* otherwise.

4.3 Cyclic admissibility and the $18 \oplus 3$ split

For symbolic dynamics and zeta-function purposes, the relevant notion of closure is *cyclic*: a length-6 word corresponds to a periodic orbit segment only if the wrap-around transition $w_6 \to w_1$ is also allowed. For the golden mean constraint, this wrap-around admissibility is equivalent to excluding $w_6 w_1 = 11$.

Remark 4.1 (Why cyclic closure is the π-channel)**.** *The cyclic closure condition is the one compatible with periodic-orbit counting and the Artin–Mazur zeta identities used in Section 5. Endpoint closure ($w_1 = w_6$) is also natural from a chain-boundary viewpoint, but it does not encode periodic admissibility for the golden mean constraint.*

Definition 4.2 (Cyclic admissibility and boundary states)**.** *Define*

$$X_6^{\mathrm{cyc}} := \{w \in X_6 :\ w_6 w_1 \neq 11\}, \qquad X_6^{\mathrm{bdry}} := X_6 \setminus X_6^{\mathrm{cyc}}.$$

We call elements of X_6^{cyc} cyclically admissible*, and elements of X_6^{bdry}* boundary states*.*

Proposition 4.3 (The boundary set has size 3)**.** *One has*

$$X_6^{\mathrm{bdry}} = \{100001,\ 100101,\ 101001\}, \qquad |X_6^{\mathrm{bdry}}| = 3, \qquad |X_6^{\mathrm{cyc}}| = 18.$$

Proof. The wrap-around transition is forbidden if and only if $w_1 = w_6 = 1$. If $w_1 = w_6 = 1$ and $w \in X_6$, then $w_2 = w_5 = 0$. The middle block (w_3, w_4) can be $(0, 0)$, $(0, 1)$, or $(1, 0)$ (but not $(1, 1)$). This yields exactly three boundary words:

$$100001, \quad 100101, \quad 101001.$$

Hence $|X_6^{\mathrm{bdry}}| = 3$ and $|X_6^{\mathrm{cyc}}| = |X_6| - 3 = 21 - 3 = 18$. □

4.4 General length-n counts and the Lucas trace

The cyclic/boundary split is not special to length 6; it admits closed formulas at every length.

Proposition 4.4 (Cyclic words, boundary words, and a Lucas-number identity). *Let* $X_n \subset \{0,1\}^n$ *be the set of length-n words with no adjacent ones, and define*

$$X_n^{\mathrm{cyc}} := \{w \in X_n : \ w_n w_1 \neq 11\}, \qquad X_n^{\mathrm{bdry}} := X_n \setminus X_n^{\mathrm{cyc}}.$$

Then for $n \geq 2$*,*

$$|X_n^{\mathrm{bdry}}| = F_{n-2}, \qquad |X_n^{\mathrm{cyc}}| = |X_n| - |X_n^{\mathrm{bdry}}| = F_{n+2} - F_{n-2}.$$

Moreover, if $A = \left(\begin{smallmatrix} 1 & 1 \\ 1 & 0 \end{smallmatrix}\right)$*, then*

$$|X_n^{\mathrm{cyc}}| = \mathrm{Tr}(A^n) = F_{n-1} + F_{n+1},$$

which equals the Lucas number L_n *(defined by* $L_0 = 2$*,* $L_1 = 1$*, and* $L_{n+2} = L_{n+1} + L_n$*).*

Proof. A word $w \in X_n$ lies in X_n^{bdry} if and only if $w_1 = w_n = 1$. Then admissibility forces $w_2 = w_{n-1} = 0$, and the middle block $(w_3, \ldots, w_{n-2})$ is an arbitrary admissible word of length $n-4$. Thus $|X_n^{\mathrm{bdry}}| = |X_{n-4}| = F_{(n-4)+2} = F_{n-2}$. The identity $|X_n^{\mathrm{cyc}}| = |X_n| - |X_n^{\mathrm{bdry}}|$ is immediate.

For the trace formula, cyclic admissible words of length n are in bijection with closed walks of length n on the Markov graph with adjacency matrix A, hence their number is $\mathrm{Tr}(A^n)$; see, e.g., [4]. To compute $\mathrm{Tr}(A^n)$, extend Fibonacci by $F_0 := 0$ and note the matrix identity

$$A^n = \begin{pmatrix} F_{n+1} & F_n \\ F_n & F_{n-1} \end{pmatrix} \qquad (n \geq 1),$$

which follows by induction using $A^{n+1} = A^n A$ and the Fibonacci recursion. Taking the trace yields $\mathrm{Tr}(A^n) = F_{n+1} + F_{n-1}$. ☐

Remark 4.5 (Two natural π-defects). *The* π*-channel admits two closely related nonnegative defects on* X_6*:*

- ***Endpoint defect:*** $\mathcal{D}_\pi^{\mathrm{end}}(w) := \mathbf{1}_{\{w_1 \neq w_6\}}$.
- ***Cyclic defect:*** $\mathcal{D}_\pi^{\mathrm{cyc}}(w) := \mathbf{1}_{\{w_1 = w_6 = 1\}}$.

The cyclic defect is the one compatible with periodic-orbit counting and zeta identities (Section 5), and it is responsible for the canonical $18 \oplus 3$ *split within* X_6*.*

5 The e-channel: Abel–zeta stability and a pole barrier

The e-channel formalizes an *analytic stability* viewpoint: stable grammars admit generating functions with a controlled holomorphic domain. In the finite-state setting of shifts of finite type, the relevant analytic content is completely captured by a rational zeta function and its pole locations (equivalently, the spectral radius of the adjacency matrix). We use the variable r in the unit disk as a normalized power-series parameter; no Abel finite-part subtleties are required at this level.

5.1 Artin–Mazur zeta for the golden mean shift

Let $A = \left(\begin{smallmatrix}1&1\\1&0\end{smallmatrix}\right)$ be the golden mean adjacency matrix (Section 4). For $n \geq 1$, let P_n be the number of period-n points of the golden mean shift. Equivalently, P_n is the number of cyclic admissible words of length n (wrap-around allowed).

Proposition 5.1 (Determinant identity for the zeta function)**.** *Define the Artin–Mazur zeta function*

$$\zeta(z) := \exp\Big(\sum_{n\geq 1} \frac{P_n}{n} z^n\Big).$$

Then

$$\zeta(z) = \frac{1}{\det(I - zA)} = \frac{1}{1 - z - z^2}.$$

Proof. The zeta function ζ was introduced in [14]. For a topological Markov shift with adjacency matrix A, one has $P_n = \mathrm{Tr}(A^n)$ and

$$\zeta(z) = \exp\Big(\sum_{n\geq 1} \frac{\mathrm{Tr}(A^n)}{n} z^n\Big) = \frac{1}{\det(I - zA)};$$

see, e.g., [4, 7]. For $A = \left(\begin{smallmatrix}1&1\\1&0\end{smallmatrix}\right)$, a direct computation gives $\det(I - zA) = 1 - z - z^2$. $\square$

Remark 5.2 (Rationality for finite-state shifts)**.** *For shifts of finite type given by a finite adjacency matrix, the determinant identity implies that ζ is a rational function whose poles are reciprocals of eigenvalues of the matrix. In particular, analytic stability questions reduce to spectral information (Perron–Frobenius theory) in the finite-state setting [4, 7].*

5.2 Pole locations and spectral radius

The eigenvalues of A are

$$\lambda_\pm = \frac{1 \pm \sqrt{5}}{2} = \varphi,\ -\varphi^{-1}.$$

Hence $\zeta(z)$ has poles at $z = \varphi^{-1}$ and $z = -\varphi$. The nearest singularity to the origin is at $z = \varphi^{-1}$, so the radius of convergence of the defining power series is φ^{-1}.

Proposition 5.3 (Radius of convergence equals inverse spectral radius)**.** *Let A be the adjacency matrix of a shift of finite type and let $\zeta_A(z) = 1/\det(I - zA)$. Then the radius of convergence of the power series defining* $\log \zeta_A(z)$ *around $z = 0$ equals* $1/\rho(A)$.

Proof. By the determinant formula, poles of ζ_A are reciprocals of eigenvalues of A. Thus the singularity of ζ_A nearest to the origin lies at modulus $1/\rho(A)$. Since $\log \zeta_A$ is analytic wherever ζ_A is analytic and nonzero, the same radius controls the defining series for $\log \zeta_A$. $\square$

5.3 Spectral (Abel-type) normalization and a unit-disk holomorphy domain

Introduce a normalized complex variable r by setting

$$z = \frac{r}{\varphi}.$$

Define

$$\widetilde{\zeta}(r) := \zeta\Big(\frac{r}{\varphi}\Big) = \frac{1}{1 - \frac{r}{\varphi} - \frac{r^2}{\varphi^2}} = \frac{1}{(1 - r)(1 + \varphi^{-2} r)}.$$

The factorization follows from $\varphi^2 = \varphi + 1$, which implies $1 - \varphi^{-2} = \varphi^{-1}$. Here $\varphi^{-2} r$ denotes the product $\varphi^{-2} \cdot r$ (not an exponential dependence on r). Then $\widetilde{\zeta}$ is holomorphic in the open unit disk $|r| < 1$, and its principal pole is located at $r = 1$ on the boundary. The second pole $r = -\varphi^2$ lies strictly outside the unit disk.

Remark 5.4 (A pole-barrier interpretation)**.** *The change of variables $z = r/\varphi$ is a spectral normalization that places the dominant singularity of the golden mean zeta at the boundary point $r = 1$. In this coordinate the normalized zeta has no poles in $|r| < 1$. For comparison, the full two-symbol shift (allowing 11) has adjacency matrix $B = \left(\begin{smallmatrix}1&1\\1&1\end{smallmatrix}\right)$ with spectral radius $\rho(B) = 2$ and zeta $(1-2z)^{-1}$ [4]. Under the same normalization $z = r/\varphi$, its pole lies at $r = \varphi/\rho(B) = \varphi/2 < 1$. Equivalently, the full shift has a larger exponential growth rate than the golden mean shift, so its zeta becomes singular strictly inside the unit disk in the golden-mean normalization.*

Proposition 5.5 (Spectral normalization yields a unit-disk holomorphy domain)**.** *Let A be a nonnegative $d \times d$ matrix with spectral radius $\rho(A) > 0$ and define*

$$\zeta_A(z) := \frac{1}{\det(I - zA)}.$$

Then the normalized function

$$\widetilde{\zeta}_A(r) := \zeta_A\Big(\frac{r}{\rho(A)}\Big) = \frac{1}{\det\Big(I - \frac{r}{\rho(A)}A\Big)}$$

is holomorphic on $|r| < 1$. Moreover, if A has an eigenvalue equal to $\rho(A)$ (as in Perron–Frobenius theory), then $r = 1$ is a pole on the boundary.

Proof. Poles of ζ_A are reciprocals of eigenvalues of A. Thus poles of $\widetilde{\zeta}_A$ occur at $r = \rho(A)/\lambda$, where λ ranges over eigenvalues. Since $|\lambda| \le \rho(A)$ for all eigenvalues, every pole satisfies $|r| \ge 1$, proving holomorphy on $|r| < 1$. If $\lambda = \rho(A)$ is an eigenvalue, then $r = 1$ is a pole. □

5.4 A genuinely distinct e-channel via weighted shifts

The minimal window-6 model identifies the e-defect with a local forbidden-event counter. To obtain an analytic stability channel that is *not* equivalent to a single forbidden word, one may keep the underlying grammar fixed but introduce a weight (a potential) and study the associated weighted zeta / transfer-operator spectrum. This is standard in thermodynamic formalism; see [5–7].

Proposition 5.6 (A one-parameter weighted golden mean zeta)**.** *For $\beta \in \mathbb{R}$, define a weighted adjacency matrix*

$$A_\beta := \begin{pmatrix} 1 & \mathrm{e}^{-\beta} \\ 1 & 0 \end{pmatrix}.$$

Then the associated (finite-state) zeta function is

$$\zeta_\beta(z) = \frac{1}{\det(I - zA_\beta)} = \frac{1}{1 - z - \mathrm{e}^{-\beta} z^2},$$

with spectral radius

$$\rho(A_\beta) = \frac{1 + \sqrt{1 + 4\mathrm{e}^{-\beta}}}{2}.$$

Under the fixed normalization $z = r/\varphi$, the normalized zeta has a pole in $|r| < 1$ if and only if $\rho(A_\beta) > \varphi$ (equivalently $\beta < 0$).

Proof. The determinant identity is a direct computation:

$$\det(I - zA_\beta) = \det\begin{pmatrix} 1 - z & -z\mathrm{e}^{-\beta} \\ -z & 1 \end{pmatrix} = (1 - z) - \mathrm{e}^{-\beta} z^2.$$

The eigenvalues of A_β are the roots of $\lambda^2 - \lambda - \mathrm{e}^{-\beta} = 0$, hence the spectral radius is the positive root. Under $z = r/\varphi$, the closest pole to the origin in the r-plane is located at $r = \varphi/\rho(A_\beta)$, which lies inside $|r| < 1$ exactly when $\rho(A_\beta) > \varphi$. □

Remark 5.7 (Channel separation)**.** *The φ-constraint may still be formulated as a pure grammar restriction (e.g. forbidding* 11*), while the weighted* e*-channel above changes analytic stability through $\rho(A_\beta)$ without changing the forbidden word set. At finite window length m, a natural local observable for this weighted channel is the Hamming weight $|w|_1$ (or edge-count observables such as the number of $0 \to 1$ transitions), which is generally not equal to the φ-defect counting forbidden adjacent pairs.*

Remark 5.8 (Connection to pressure and transfer operators)**.** *For shifts of finite type and locally constant potentials, thermodynamic formalism identifies the topological pressure with the logarithm of the leading eigenvalue of a weighted transfer operator. In the finite-state setting above, this leading eigenvalue is precisely the spectral radius $\rho(A_\beta)$, so $\log \rho(A_\beta)$ is the pressure of the corresponding potential (see [5, 6]). Accordingly, pole locations of ζ_β encode pressure via $\rho(A_\beta)$, and the normalization $z = r/\rho(A)$ is a concrete finite-state manifestation of an Abel-type parameter $r \uparrow 1$ approaching the dominant spectral singularity.*

5.5 A finite-window e-defect and its coincidence with the phi-constraint

Define the local defect

$$\mathcal{D}_{\mathrm{e}}(w) := \#\{\, i \in \{1, \dots, 5\} : \ w_i = w_{i+1} = 1 \,\}, \qquad w \in \Omega_6.$$

This counts the number of forbidden adjacent excitations inside the length-6 window.

Proposition 5.9 (e-stability coincides with φ-legality at window length 6)**.** *For $w \in \Omega_6$ one has $\mathcal{D}_{\mathrm{e}}(w) = 0$ if and only if $w \in X_6$. Equivalently,*

$$\{w \in \Omega_6 : \ \mathcal{D}_{\mathrm{e}}(w) = 0\} = X_6.$$

Proof. By definition $\mathcal{D}_{\mathrm{e}}(w) = 0$ exactly when no adjacent pair (w_i, w_{i+1}) equals $(1, 1)$, which is precisely the definition of X_6. □

Remark 5.10 (Minimal-model degeneracy)**.** *At the level of a length-*6 *local model, the φ-channel (forbidden word grammar) and the* e*-channel (analytic stability interpreted via forbidden events) coincide as constraints on Ω_6. Their distinction is semantic: φ refers to Fibonacci growth / Zeckendorf legality, while* e *refers to holomorphy domains and pole barriers of zeta/Abel transforms. Nontrivial separation of the channels can emerge in higher-resolution models (longer windows, weighted shifts, or infinite-state extensions), where analytic stability is no longer captured by a single local forbidden law.*

6 A computable folding map: Zeckendorf normalization and window truncation

The φ-constraint selects the admissible set $X_6 \subset \Omega_6$ of size 21. To obtain a genuine *folding* from 64 microstates to these 21 stable types, we define an explicit surjection using Zeckendorf representations and finite windows.

6.1 Zeckendorf base and uniqueness

Let $(F_n)_{n \geq 1}$ denote the Fibonacci numbers with $F_1 = 1$, $F_2 = 1$, and $F_{n+2} = F_{n+1} + F_n$. In Zeckendorf representation one uses the Fibonacci weights $F_2, F_3, F_4, \dots$, i.e. the weight attached to digit c_k is F_{k+1}.

Lemma 6.1 (Binet formula and standard Fibonacci bounds). *Let* $\varphi = \frac{1+\sqrt{5}}{2}$ *and* $\psi = \frac{1-\sqrt{5}}{2} = -\varphi^{-1}$*. Then for all* $n \geq 1$*,*

$$F_n = \frac{\varphi^n - \psi^n}{\sqrt{5}}.$$

In particular, for all $n \geq 2$ *one has the two-sided bounds*

$$\varphi^{n-2} \leq F_n \leq \varphi^{n-1},$$

and hence $F_n = \Theta(\varphi^n)$*.*

Proof. This is classical; see, e.g., [15]. For completeness, Appendix F records a short proof of the stated two-sided bounds. □

Theorem 6.2 (Zeckendorf uniqueness). *Every integer* $N \in \mathbb{N}_{>0}$ *admits a unique representation*

$$N = \sum_{k=1}^{m} c_k F_{k+1}, \qquad c_k \in \{0,1\}, \qquad c_k c_{k+1} = 0.$$

Proof. See [13]; modern accounts are available in [8, 9]. □

For convenience, set $\mathrm{Z}(0) := (0,0,0,\dots)$ and for $N > 0$ let $\mathrm{Z}(N) := (c_1, c_2, \dots)$ denote the Zeckendorf digit sequence of N (extended by zeros).

6.2 Window projection and the folding map Fold_6

For $m \geq 1$ define the window projection

$$\pi_m(\mathrm{Z}(N)) := (c_1, \dots, c_m) \in \{0,1\}^m.$$

By Theorem 6.2, $\pi_m(\mathrm{Z}(N))$ always has no adjacent ones, so $\pi_m(\mathrm{Z}(N)) \in X_m$.

Definition 6.3 (Resolution folding map at window length 6). *Define*

$$\mathrm{Fold}_6 : \{0, 1, \dots, 63\} \to X_6, \qquad \mathrm{Fold}_6(N) := \pi_6(\mathrm{Z}(N)).$$

Definition 6.4 (Folding of 6-bit microstates). *Using the binary indexing map* $\mathrm{int}_6 : \Omega_6 \to \{0, \dots, 63\}$ *from Section 2, define*

$$\mathsf{Fold}_6^{\mathrm{bin}} : \Omega_6 \to X_6, \qquad \mathsf{Fold}_6^{\mathrm{bin}}(w) := \mathrm{Fold}_6(\mathrm{int}_6(w)).$$

Proposition 6.5 (Surjectivity onto the 21 admissible types). *The image of* Fold_6 *is exactly* X_6*; in particular* Fold_6 *is surjective and* $|\mathrm{Im}(\mathrm{Fold}_6)| = 21$*.*

Proof. For every N, the Zeckendorf digits have no adjacent ones, hence $\mathrm{Fold}_6(N) \in X_6$ and $\mathrm{Im}(\mathrm{Fold}_6) \subseteq X_6$.

Conversely, let $w = (w_1, \dots, w_6) \in X_6$. Define

$$N_w := \sum_{k=1}^{6} w_k F_{k+1}.$$

Since w has no adjacent ones, the digit string $(w_1, \dots, w_6, 0, 0, \dots)$ is already Zeckendorf-admissible, hence it equals $\mathrm{Z}(N_w)$ by uniqueness. Therefore $\mathrm{Fold}_6(N_w) = w$, so $X_6 \subseteq \mathrm{Im}(\mathrm{Fold}_6)$. □

6.3 A canonical bijection on the Zeckendorf range

Let

$$V(w) := \sum_{k=1}^{6} w_k F_{k+1} \qquad (w \in X_6)$$

be the Zeckendorf value of a length-6 admissible word.

Lemma 6.6 (Maximal Zeckendorf value at window length m)**.** *For every $m \geq 1$ one has*

$$\max_{w \in X_m} \sum_{k=1}^{m} w_k F_{k+1} = F_{m+2} - 1.$$

In particular, for $m = 6$ one has $\max_{w \in X_6} V(w) = F_8 - 1 = 20$.

Proof. Let $M_n := \max_{w \in X_n} \sum_{k=1}^{n} w_k F_{k+1}$ for $n \geq 1$. If $w_n = 0$, the best contribution is M_{n-1}. If $w_n = 1$, then $w_{n-1} = 0$ and the best contribution is $F_{n+1} + M_{n-2}$. Hence $M_n = \max\{M_{n-1}, F_{n+1} + M_{n-2}\}$ with initial values $M_1 = F_2 = 1$ and $M_2 = F_3 = 2$. Using $F_{n+3} = F_{n+2} + F_{n+1}$, one checks by induction that $M_n = F_{n+2} - 1$. Setting $n = m$ yields $M_m = F_{m+2} - 1$. □

Proposition 6.7 (General window length: canonical bijection on the Zeckendorf range)**.** *For $m \geq 1$, define*

$$\mathrm{Fold}_m : \{0, 1, \ldots, 2^m - 1\} \to X_m, \qquad \mathrm{Fold}_m(N) := \pi_m(\mathrm{Z}(N)).$$

Then Fold_m is surjective, and it restricts to a bijection

$$\mathrm{Fold}_m : \{0, 1, \ldots, F_{m+2} - 1\} \xrightarrow{\sim} X_m.$$

Proof. Surjectivity follows as in Proposition 6.5: for $w \in X_m$, the integer $N_w := \sum_{k=1}^{m} w_k F_{k+1}$ satisfies $\mathrm{Fold}_m(N_w) = w$. Lemma 6.6 implies $0 \leq N_w \leq F_{m+2} - 1$.

For the bijection claim, Lemma 6.6 implies that every integer $N \in \{0, \ldots, F_{m+2} - 1\}$ has Zeckendorf digits supported within $\{1, \ldots, m\}$, so $\mathrm{Fold}_m(N)$ equals the full digit string of N and therefore determines N uniquely by Zeckendorf uniqueness. □

Remark 6.8 (Average degeneracy for the $2^m \to F_{m+2}$ compression)**.** *On the full index set $\{0, \ldots, 2^m - 1\}$, the average preimage size of Fold_m is*

$$\frac{2^m}{|X_m|} = \frac{2^m}{F_{m+2}} \sim \sqrt{5}\, \varphi^{-2} \Big(\frac{2}{\varphi}\Big)^m,$$

Thus, while the stable type count grows as $F_{m+2} \asymp \varphi^m$, the mean folding degeneracy grows exponentially with base $2/\varphi > 1$. Theorem 6.15 gives a complete closed-form degeneracy classification in the base case $m = 6$.

Proposition 6.9 (Exact closed form and an error bound for the mean degeneracy)**.** *Define the mean preimage size*

$$\bar{d}_m := \frac{1}{|X_m|} \sum_{w \in X_m} |\mathrm{Fold}_m^{-1}(w)| = \frac{2^m}{|X_m|} = \frac{2^m}{F_{m+2}}.$$

Then for every $m \geq 1$,

$$\bar{d}_m \Big(\frac{\varphi}{2}\Big)^m = \frac{\sqrt{5}}{\varphi^2} \cdot \frac{1}{1 - (-\varphi^{-2})^{m+2}}.$$

In particular,

$$\left| \bar{d}_m \Big(\frac{\varphi}{2}\Big)^m - \frac{\sqrt{5}}{\varphi^2} \right| \leq \frac{\sqrt{5}}{\varphi^2} \cdot \frac{\varphi^{-2m-4}}{1 - \varphi^{-4}}.$$

Proof. By Lemma 6.1 (Binet formula) and $\psi = -\varphi^{-1}$ one has

$$F_{m+2} = \frac{\varphi^{m+2} - \psi^{m+2}}{\sqrt{5}} = \frac{\varphi^{m+2}}{\sqrt{5}}\Big(1 - \Big(\frac{\psi}{\varphi}\Big)^{m+2}\Big) = \frac{\varphi^{m+2}}{\sqrt{5}}\big(1 - (-\varphi^{-2})^{m+2}\big).$$

Substituting into $\overline{d}_m = 2^m/F_{m+2}$ yields the stated closed form. For the bound, set $x := (-\varphi^{-2})^{m+2}$, so $|x| = \varphi^{-2m-4}$ and

$$\left|\frac{1}{1-x} - 1\right| = \frac{|x|}{|1-x|} \le \frac{|x|}{1-|x|} \le \frac{\varphi^{-2m-4}}{1-\varphi^{-4}},$$

since $|x| \le \varphi^{-4}$ for $m \ge 0$. □

Remark 6.10 (Origin of the constant $\sqrt{5}/\varphi^2$)**.** *At the level of first-order asymptotics, Binet's formula gives $F_{m+2} \sim \varphi^{m+2}/\sqrt{5}$, hence $\overline{d}_m = 2^m/F_{m+2} \sim \sqrt{5}\,\varphi^{-2}(2/\varphi)^m$ and therefore $\overline{d}_m(\varphi/2)^m \to \sqrt{5}/\varphi^2$.*

Proposition 6.11 (General preimages for Fold_m)**.** *Fix $m \ge 1$ and $w = (w_1, \dots, w_m) \in X_m$. Define its Zeckendorf value*

$$V_m(w) := \sum_{k=1}^{m} w_k F_{k+1}.$$

Let $K(m)$ be the unique integer such that

$$F_{K(m)+1} \le 2^m - 1 < F_{K(m)+2}.$$

Then the full preimage set of w under Fold_m *is*

$$\mathrm{Fold}_m^{-1}(w) = \left\{ V_m(w) + \sum_{k=m+1}^{K(m)} c_k F_{k+1} : \begin{array}{l} (c_{m+1}, \dots, c_{K(m)}) \in \{0,1\}^{K(m)-m}, \\ c_k c_{k+1} = 0 \text{ for } m+1 \le k < K(m), \\ w_m c_{m+1} = 0, \\ V_m(w) + \sum_{k=m+1}^{K(m)} c_k F_{k+1} \le 2^m - 1 \end{array} \right\}.$$

In particular, the degeneracy $|\mathrm{Fold}_m^{-1}(w)|$ depends on w only through $V_m(w)$ and the boundary bit w_m.

Proof. Let $N \in \{0, \dots, 2^m - 1\}$ and write its Zeckendorf digits $\mathrm{Z}(N) = (c_1, c_2, \dots)$. By definition $\mathrm{Fold}_m(N) = w$ if and only if $(c_1, \dots, c_m) = (w_1, \dots, w_m)$. For $N \le 2^m - 1$, no Fibonacci weight above $F_{K(m)+1}$ can appear in $\mathrm{Z}(N)$, hence $c_k = 0$ for all $k > K(m)$. Therefore

$$N = V_m(w) + \sum_{k=m+1}^{K(m)} c_k F_{k+1}$$

with the adjacency constraints $c_k c_{k+1} = 0$ and the boundary constraint $w_m c_{m+1} = 0$. Finally, $N \in \{0, \dots, 2^m - 1\}$ is equivalent to the stated inequality bound on the tail sum. □

Corollary 6.12 (Uniform upper bound and extremizer)**.** *With $K(m)$ as in Proposition 6.11, one has for every $w \in X_m$:*

$$|\mathrm{Fold}_m^{-1}(w)| \le |X_{K(m)-m}| = F_{K(m)-m+2}.$$

Moreover, $|\mathrm{Fold}_m^{-1}(w)|$ is nonincreasing in $V_m(w)$, and it is maximized at $w = 0^m$.

Proof. Dropping the inequality constraint in Proposition 6.11 yields an injection

$$\mathrm{Fold}_m^{-1}(w) \hookrightarrow \{(c_{m+1}, \ldots, c_{K(m)}) \in \{0,1\}^{K(m)-m} : \ c_k c_{k+1} = 0\},$$

whose cardinality is $|X_{K(m)-m}| = F_{K(m)-m+2}$. Monotonicity in $V_m(w)$ holds because increasing $V_m(w)$ decreases the available tail budget $2^m - 1 - V_m(w)$, hence can only remove admissible tails. The additional boundary restriction $w_m c_{m+1} = 0$ is weakest when $w_m = 0$. Thus the maximal degeneracy is attained for the smallest value $V_m(w)$ with $w_m = 0$, namely $w = 0^m$. □

Proposition 6.13 (Location of the Zeckendorf cutoff $K(m)$)**.** *Let $K(m)$ be the unique integer such that $F_{K(m)+1} \le 2^m - 1 < F_{K(m)+2}$. Then for every $m \ge 2$,*

$$(m-1)\log_\varphi 2 - 1 \ \le \ K(m) \ < \ m \log_\varphi 2 + 1.$$

In particular, $K(m) = m\log_\varphi 2 + O(1)$ and

$$K(m) - m = (\log_\varphi 2 - 1)m + O(1) = \log_\varphi\Big(\frac{2}{\varphi}\Big)m + O(1).$$

Proof. Since $m \ge 2$ one has $2^{m-1} \le 2^m - 1 < 2^m$. From $2^m - 1 < F_{K(m)+2}$ and Lemma 6.1 (upper bound) we get

$$2^{m-1} \le 2^m - 1 < F_{K(m)+2} \le \varphi^{K(m)+1},$$

so $(m-1)\log_\varphi 2 \le K(m)+1$ and hence $K(m) \ge (m-1)\log_\varphi 2 - 1$. From $F_{K(m)+1} \le 2^m - 1 < 2^m$ and Lemma 6.1 (lower bound) we get

$$\varphi^{K(m)-1} \le F_{K(m)+1} \le 2^m,$$

so $K(m) - 1 \le m\log_\varphi 2$ and hence $K(m) < m\log_\varphi 2 + 1$. The asymptotic statements follow. □

Corollary 6.14 (Maximal degeneracy is of the same exponential order as the mean)**.** *Let*

$$\overline{d}_m := \frac{1}{|X_m|} \sum_{w \in X_m} |\mathrm{Fold}_m^{-1}(w)| = \frac{2^m}{|X_m|} = \frac{2^m}{F_{m+2}}$$

be the mean preimage size, and let $d_m^{\max} := \max_{w\in X_m} |\mathrm{Fold}_m^{-1}(w)|$. Then for all $m \ge 2$,

$$\varphi^{-1}\Big(\frac{2}{\varphi}\Big)^m \ \le \ \overline{d}_m \ \le \ d_m^{\max} \ \le \ \varphi^2\Big(\frac{2}{\varphi}\Big)^m.$$

In particular, $d_m^{\max} = \Theta\big((2/\varphi)^m\big)$ and $d_m^{\max} \le \varphi^3\,\overline{d}_m$.

Proof. The lower bound $\overline{d}_m \le d_m^{\max}$ is trivial. By Lemma 6.1 (upper bound) one has $F_{m+2} \le \varphi^{m+1}$, hence

$$\overline{d}_m = \frac{2^m}{F_{m+2}} \ge \frac{2^m}{\varphi^{m+1}} = \varphi^{-1}\Big(\frac{2}{\varphi}\Big)^m.$$

For the upper bound on $d_m^{\max}$, Corollary 6.12 gives

$$d_m^{\max} \le F_{K(m)-m+2} \le \varphi^{K(m)-m+1} = \frac{\varphi^{K(m)+1}}{\varphi^m}.$$

Lemma 6.1 (lower bound) implies $\varphi^{K(m)-1} \le F_{K(m)+1} \le 2^m$, hence $\varphi^{K(m)+1} \le \varphi^2 2^m$. Substituting yields $d_m^{\max} \le \varphi^2(2/\varphi)^m$. Finally, Lemma 6.1 (lower bound) implies $F_{m+2} \ge \varphi^m$, so $(2/\varphi)^m \le \varphi\,\overline{d}_m$ and therefore $d_m^{\max} \le \varphi^3\,\overline{d}_m$. □

m	2^m	F_{m+2}	$\overline{d}_m = \frac{2^m}{F_{m+2}}$	$\overline{d}_m \left(\frac{\varphi}{2}\right)^m$
2	4	3	1.333333	0.872678
3	8	5	1.600000	0.847214
4	16	8	2.000000	0.856763
5	32	13	2.461538	0.853090
6	64	21	3.047619	0.854489
7	128	34	3.764706	0.853954
8	256	55	4.654545	0.854158
9	512	89	5.752809	0.854080
10	1024	144	7.111111	0.854110
11	2048	233	8.789700	0.854099
12	4096	377	10.864721	0.854103

Table 2: Quantitative scaling of mean folding degeneracy. Proposition 6.9 gives a closed form and an explicit error bound for the normalized quantity $\overline{d}_m(\varphi/2)^m$, whose limit equals $\sqrt{5}/\varphi^2$. Table rows are generated by `scripts/exp_foldm_scaling_table.py` into `sections/generated/foldm_scaling_rows.tex`.

6.4 Strong quantitative preimage structure on the index set $\{0, \dots, 63\}$

The folding $\mathrm{Fold}_6 : \{0, \dots, 63\} \to X_6$ is many-to-one because $2^6 = 64$ exceeds $|X_6| = F_8 = 21$. Nevertheless, its preimage structure is completely explicit at window length 6.

Theorem 6.15 (Preimage classification and degeneracy histogram for Fold_6)**.** *For $w \in X_6$, let $V(w) = \sum_{k=1}^{6} w_k F_{k+1}$. Then the preimage set $\mathrm{Fold}_6^{-1}(w) \subset \{0, \dots, 63\}$ is:*

$$\mathrm{Fold}_6^{-1}(w) = \begin{cases} \{V(w),\, V(w) + F_9\}, & w_6 = 1, \\ \{V(w),\, V(w) + F_8,\, V(w) + F_9,\, V(w) + F_{10}\}, & w_6 = 0 \text{ and } V(w) \le 8, \\ \{V(w),\, V(w) + F_8,\, V(w) + F_9\}, & w_6 = 0 \text{ and } V(w) > 8. \end{cases}$$

In particular, $|\mathrm{Fold}_6^{-1}(w)| \in \{2, 3, 4\}$ for all $w \in X_6$, and the degeneracy histogram is

$$\begin{aligned} \#\{w \in X_6 :\ |\mathrm{Fold}_6^{-1}(w)| = 2\} &= 8, \\ \#\{w \in X_6 :\ |\mathrm{Fold}_6^{-1}(w)| = 3\} &= 4, \\ \#\{w \in X_6 :\ |\mathrm{Fold}_6^{-1}(w)| = 4\} &= 9. \end{aligned}$$

Remark 6.16 (Sanity check: total mass and mean degeneracy)**.** *The histogram in Theorem 6.15 implies*

$$2 \cdot 8 + 3 \cdot 4 + 4 \cdot 9 = 64,$$

which matches $|\{0, \dots, 63\}|$. Dividing by $|X_6| = 21$ yields the mean degeneracy $\overline{d}_6 = 64/21$, consistent with Proposition 6.9.

Proof. For $N \le 63$, the Fibonacci weight $F_{11} = 89$ is too large, so the Zeckendorf digits satisfy $c_k = 0$ for all $k \ge 10$. Thus only the digits c_7, c_8, c_9 (with weights $F_8 = 21$, $F_9 = 34$, $F_{10} = 55$) can appear above the window.

Fix $w \in X_6$ and suppose $\mathrm{Fold}_6(N) = w$. Then the first six Zeckendorf digits of N equal w, hence

$$N = V(w) + c_7 F_8 + c_8 F_9 + c_9 F_{10}, \qquad c_7, c_8, c_9 \in \{0, 1\},$$

with the adjacency constraints $w_6 c_7 = 0$, $c_7 c_8 = 0$, and $c_8 c_9 = 0$.

Case 1: $w_6 = 1$. Then $c_7 = 0$. The admissible choices for (c_8, c_9) are $(0, 0)$ and $(1, 0)$ and $(0, 1)$. However, $V(w) \geq F_7 = 13$ in this case, hence $V(w) + F_{10} > 63$ and the option $(c_8, c_9) = (0, 1)$ is impossible under the bound $N \leq 63$. Therefore $\mathrm{Fold}_6^{-1}(w) = \{V(w), V(w) + F_9\}$.

Case 2: $w_6 = 0$. Now c_7 may be 0 or 1, but $(c_7, c_8) = (1, 1)$ is forbidden. Moreover, if $c_9 = 1$ then $c_8 = 0$, and necessarily $c_7 = 0$ as well since $F_{10} + F_8 = 55 + 21 > 63$. Thus the only admissible offsets are 0, F_8, F_9, and possibly F_{10}. The offset F_{10} is admissible if and only if $V(w) \leq 63 - F_{10} = 8$. This yields the stated preimage formulas.

Degeneracy counts. There are exactly 8 admissible words in X_6 with $w_6 = 1$ (equivalently, length-4 admissible prefixes followed by 01), hence 8 outputs of degeneracy 2. Among the $w_6 = 0$ outputs, $V(w) > 8$ holds precisely when $w_5 = 1$ and $(w_1, w_2, w_3) \neq (0, 0, 0)$ (since $w_4 = 0$ is forced by admissibility), yielding 4 such words and hence 4 outputs of degeneracy 3. The remaining $21 - 8 - 4 = 9$ outputs have degeneracy 4. □

6.5 Algorithmic computation

A standard greedy algorithm computes $\mathrm{Z}(N)$ by iteratively subtracting the largest Fibonacci weight $\leq N$ while skipping adjacent indices (to enforce $c_k c_{k+1} = 0$); see, e.g., [8]. We record pseudocode and a reference pure-Python implementation in Appendix C.

6.6 Local normalization by a rewriting rule (optional viewpoint)

There exist normalization procedures that convert a non-canonical Fibonacci digit string into the Zeckendorf canonical form using local rewrites induced by the Fibonacci recursion. One concrete instance is the rewrite

$$011 \;\Rightarrow\; 100,$$

interpreted for digits written in *descending* weight order (most significant digit on the left), since $F_n = F_{n-1} + F_{n-2}$. Iterating such rewrites (together with a finite carry protocol) yields a canonical Zeckendorf form; see, e.g., [8].

7 Resolution change as recursion: Hilbert refinement and Zeckendorf depth

The phrase "resolution" in this paper refers to two independent parameters:

- a *spatial* resolution parameter (Hilbert order n), controlling the screen lattice size $2^n \times 2^n$ and the address map H_n;
- a *syntactic* resolution parameter (Zeckendorf window length m), controlling the number of admissible types $|X_m| = F_{m+2}$.

Resolution folding couples these recursions: spatial refinement changes locality/embedding, while syntactic refinement changes the stable type space.

7.1 Hilbert recursion: $n \mapsto n + 1$

The Hilbert address map admits a self-similar recursion: the order-$n+1$ curve consists of four order-n sub-curves, one in each quadrant, with quadrant-dependent rotations/reflections. Concretely, there exist quadrant offsets $\Delta_q \in \Lambda_{n+1}$ and transforms $T_q \in D_4$ such that

$$H_{n+1}(t) = \Delta_q + T_q(H_n(t')),$$

where $t \in \{0, \dots, 2^{2(n+1)} - 1\}$ is decomposed as

$$t = q \cdot 2^{2n} + t', \qquad q \in \{0, 1, 2, 3\}, \quad t' \in \{0, \dots, 2^{2n} - 1\}.$$

This is the discrete manifestation of the Hilbert curve's classical self-similarity; see [11].

Remark 7.1 (Locality under recursion)**.** *The defining purpose of Hilbert addressing is that the one-step scan adjacency $t \sim t+1$ compiles to nearest-neighbor adjacency on the screen lattice. As n increases, this locality property is preserved by construction, while the induced neighborhood model can vary across the D_4 layout family.*

7.2 Zeckendorf recursion: $m \mapsto m + 1$

Let $X_m \subset \{0, 1\}^m$ be the set of length-m words with no adjacent ones. Proposition 3.1 implies the Fibonacci recursion

$$|X_{m+1}| = |X_m| + |X_{m-1}|.$$

Thus $|X_m| \asymp \varphi^m$, while the full word space size is 2^m. In particular, as m grows the absolute number of admissible types increases, but the admissible density $|X_m|/2^m$ decreases.

Proposition 7.2 (Recursive uplift decomposition of the admissible language)**.** *For every $m \geq 1$, define two injections*

$$U_0 : X_m \to X_{m+1}, \quad U_0(w) := w0, \qquad U_1 : X_{m-1} \to X_{m+1}, \quad U_1(v) := v01,$$

where concatenation is written by juxtaposition. Then

$$X_{m+1} = U_0(X_m) \sqcup U_1(X_{m-1}),$$

a disjoint union partitioned by the last bit (words ending in 0 versus words ending in 1).

Proof. Every word in $U_0(X_m)$ ends in 0 and is admissible because appending 0 cannot create an adjacent pair of ones. Every word in $U_1(X_{m-1})$ ends in 01 and is admissible because the inserted 0 separates any preceding digit from the terminal 1.

Conversely, let $w \in X_{m+1}$. If w ends in 0, write $w = u0$ with $u \in \{0, 1\}^m$; then $u \in X_m$ and $w = U_0(u)$. If w ends in 1, admissibility forces the last two bits to be 01, so $w = v01$ for some $v \in \{0, 1\}^{m-1}$; then $v \in X_{m-1}$ and $w = U_1(v)$. The union is disjoint because the last bit differs. □

Proposition 7.3 (Uplift as orthogonal isometries on ℓ^2)**.** *Let $\{\mathbf{e}_w : w \in X_m\}$ denote the standard orthonormal basis of $\ell^2(X_m)$. Define linear maps*

$$\widehat{U}_0 : \ell^2(X_m) \to \ell^2(X_{m+1}), \qquad \widehat{U}_0(\mathbf{e}_w) := \mathbf{e}_{w0},$$

and

$$\widehat{U}_1 : \ell^2(X_{m-1}) \to \ell^2(X_{m+1}), \qquad \widehat{U}_1(\mathbf{e}_v) := \mathbf{e}_{v01}.$$

Then $\widehat{U}_0$ and $\widehat{U}_1$ are isometries with orthogonal ranges, and one has the orthogonal direct sum decomposition

$$\ell^2(X_{m+1}) = \widehat{U}_0 \ell^2(X_m) \;\oplus\; \widehat{U}_1 \ell^2(X_{m-1}).$$

Proof. Since $\widehat{U}_0$ and $\widehat{U}_1$ map orthonormal bases to orthonormal sets, they extend to isometries. Their ranges are orthogonal because words in $U_0(X_m)$ end in 0 while words in $U_1(X_{m-1})$ end in 1. Finally, Proposition 7.2 gives a disjoint union partition of X_{m+1} into the two image sets, hence $\ell^2(X_{m+1})$ decomposes as the orthogonal sum of the corresponding coordinate subspaces. □

7.3 The $64 \to 21$ projection as a scale-local statement

At local window length $m = 6$, the φ-stable type space is X_6 and has size 21. Independently, at spatial order $n = 3$ the screen lattice has $2^{2n} = 2^6 = 64$ sites. This is the minimal resolution at which the spatial carrier (64 sites) and the local word space (64 words) match exactly.

From this viewpoint, the statement "$64 \to 21$" is *scale-local*: it asserts that a six-bit local readout admits a canonical 21-type stability sector, regardless of how the readout is embedded into a spatial neighborhood by an address family.

Definition 7.4 (Balanced coupling of spatial and syntactic resolution)**.** *We call the choice*

$$m = 2n$$

the balanced coupling *between spatial Hilbert order n and Zeckendorf window length m. Equivalently, balanced coupling is the condition $|\Omega_m| = |\Lambda_n|$.*

Remark 7.5 (Why balanced coupling is natural)**.** *Under Definition 7.4, the scan index set $\{0, \ldots, 2^{2n} - 1\}$ has the same cardinality as the local word space Ω_{2n}. Thus the binary expansion provides a canonical identification of scan indices with microstate labels, and a folding map on Ω_{2n} may be viewed as a deterministic type-labeling of the entire $2^n \times 2^n$ lattice via Hilbert addressing. This is the minimal setting in which the spatial carrier and the local readout have matching finite complexity.*

Proposition 7.6 (Hilbert index length and Fibonacci type count)**.** *At spatial order n (2D), the Hilbert scan index set has size 2^{2n} and admits a canonical identification with binary words of length $2n$. Under the no-adjacent-one constraint, the admissible subset has cardinality*

$$|X_{2n}| = F_{2n+2}.$$

In particular, for $n = 3$ one has $2^{2n} = 64$ and $|X_{2n}| = F_8 = 21$.

Proof. The cardinality $|\Lambda_n| = 2^{2n}$ implies the index set is $\{0, \ldots, 2^{2n} - 1\}$. Binary expansion identifies it with $\{0, 1\}^{2n}$. The admissible count is Proposition 3.1 with $m = 2n$. ☐

7.4 Iterability: folding across scales

There are two complementary ways to iterate the construction:

- **Fixed local window, increasing spatial resolution.** Keep $m = 6$ and increase n. The same $64 \to 21$ local folding applies to each local window extracted along the scan chain, while Hilbert recursion changes how local interactions are embedded on the screen.
- **Fixed spatial carrier class, increasing syntactic resolution.** Increase the window length m (hence the admissible type space size F_{m+2}) while viewing $64 \to 21$ as the base case $m = 6$ of a general family $2^m \to F_{m+2}$.

In both cases, the folding mechanism remains explicit and auditable: it is implemented either by the projector onto X_m or by the many-to-one map Fold_m defined via Zeckendorf window truncation.

8 A three-operator packaging and an iterability interface

We now package the φ–π–e channels as commuting nonnegative defect operators on $\mathcal{H}_6$. At the minimal window length 6, this packaging is intentionally conservative: it records what is strictly proven and isolates what becomes nontrivial only beyond the finite-window grammar.

8.1 Defect functions and diagonal operators

Define the φ-defect (local grammar violation count)

$$\mathcal{D}_\varphi(w) := \#\{\, i \in \{1,\dots,5\} : \; w_i = w_{i+1} = 1 \,\}, \qquad w \in \Omega_6.$$

Define the cyclic π-defect (wrap-around closure defect on the admissible sector)

$$\mathcal{D}_\pi(w) := \mathbf{1}_{\{w_1 = w_6 = 1\}}, \qquad w \in \Omega_6,$$

and define the *finite-window* e-defect by

$$\mathcal{D}_{\mathrm{e}}(w) := \mathcal{D}_\varphi(w).$$

Remark 8.1 (Finite-window representative versus analytic e-channel)**.** *At window length* 6*, the local forbidden-event counter $\mathcal{D}_\varphi$ already captures φ-legality, hence it can be used as a conservative finite-window representative of the* e*-channel. The analytically meaningful* e*-channel discussed in Section 5 is encoded by zeta/transfer-operator spectra (and, in weighted extensions, by the dependence of $\rho(A_\beta)$ on a potential), and it is not naturally a diagonal defect on Ω_6.*

Each defect defines a diagonal self-adjoint operator on $\mathcal{H}_6$ by

$$(\widehat{\mathcal{D}}_\star \psi)(w) := \mathcal{D}_\star(w)\, \psi(w), \qquad \star \in \{\varphi, \pi, \mathrm{e}\}.$$

They commute because they are diagonal in the same basis. They also induce quadratic forms (constraint seminorms)

$$\|\psi\|_\star^2 := \langle \psi, \widehat{\mathcal{D}}_\star \psi \rangle = \sum_{w \in \Omega_6} \mathcal{D}_\star(w)\, |\psi(w)|^2.$$

8.2 Stable sector and the 21-dimensional theorem

Let $X_6 \subset \Omega_6$ be the admissible set (Section 3).

Theorem 8.2 (Stable sector dimension at window length 6)**.** *One has*

$$\ker(\widehat{\mathcal{D}}_\varphi) = \ker(\widehat{\mathcal{D}}_{\mathrm{e}}) = \ell^2(X_6), \qquad \dim \ell^2(X_6) = 21.$$

Proof. By Proposition 5.9, $\mathcal{D}_\varphi(w) = 0$ if and only if $w \in X_6$, and $\mathcal{D}_{\mathrm{e}} = \mathcal{D}_\varphi$ at this window length. Hence the kernels equal $\ell^2(X_6)$. The dimension statement is Corollary 3.2. □

8.3 The canonical 18+3 split inside the stable sector

Let $X_6^{\mathrm{cyc}}, X_6^{\mathrm{bdry}}$ be as in Proposition 4.3. Then

$$\ell^2(X_6) = \ell^2(X_6^{\mathrm{cyc}}) \;\oplus\; \ell^2(X_6^{\mathrm{bdry}}), \qquad \dim \ell^2(X_6^{\mathrm{cyc}}) = 18, \quad \dim \ell^2(X_6^{\mathrm{bdry}}) = 3,$$

and $\widehat{\mathcal{D}}_\pi$ vanishes on $\ell^2(X_6^{\mathrm{cyc}})$ while acting as the identity on $\ell^2(X_6^{\mathrm{bdry}})$.

8.4 An iterability interface: the 1+21+21+21 organization

The arithmetic identity $64 = 1 + 21 + 21 + 21$ suggests a convenient organizational template for iterating the folding architecture across scales. On $\mathcal{H}_6$ itself, any statement of the form "there exists an orthogonal $1 \oplus 21 \oplus 21 \oplus 21$ decomposition" is purely linear-algebraic and carries no channel content. What matters is whether such a decomposition can be realized *naturally* by higher-resolution constraints in which the three channels become genuinely distinct.

Definition 8.3 (A natural $1 \oplus 21 \oplus 21 \oplus 21$ interface)**.** *Consider a hierarchy of microstate spaces Ω_m and associated Hilbert spaces $\mathcal{H}_m = \ell^2(\Omega_m)$. We say that a $1 \oplus 21 \oplus 21 \oplus 21$ organization is* natural *if there exist, for a distinguished scale (or in an inductive limit), orthogonal projections*

$$P_0,\ P_\varphi,\ P_\pi,\ P_{\mathrm{e}} : \mathcal{H} \to \mathcal{H}$$

such that:

- $\mathrm{rank}(P_0) = 1$ *and* P_0 *is canonical (e.g. the projection onto the normalized uniform vector);*
- $\mathrm{rank}(P_\varphi) = \mathrm{rank}(P_\pi) = \mathrm{rank}(P_{\mathrm{e}}) = 21$*;*
- *each projector is defined by, or functorially determined from, its channel data (defect operators, cyclic closure, or zeta/transfer-operator spectra), rather than by an arbitrary basis partition;*
- *the construction is covariant under the dihedral layout symmetries (Section 2) and compatible with the uplift/recursion maps relating resolutions (Section 7).*

Remark 8.4 (Status at window length 6)**.** *At the minimal window length, the strictly proven content is Theorem 8.2 and the $18 \oplus 3$ split. Moreover, at $m = 6$ the finite-window* e*-defect coincides with the φ-defect (Section 5). Thus Definition 8.3 should be viewed as an* iterability interface *for higher-resolution regimes (longer windows, weighted transfer operators, or infinite-state extensions) where the three channels can become independent.*

9 Application interfaces (non-premise): genetics and a three-factor analogy

This section records two *interfaces*: mappings from external structures into the present $64 \to 21$ folding framework. They are not used as inputs to proofs. The purpose is to turn informal analogies into falsifiable mathematical questions (existence of low-complexity encodings, distributional invariants, and robustness under symmetry actions).

9.1 Genetic code as a 64-to-(20+Stop) compression template

Let Codon be the set of 64 nucleotide triplets over a four-letter alphabet, and let

$$\mathsf{Gen} : \mathsf{Codon} \twoheadrightarrow \mathsf{AA} \cup \{\mathrm{Stop}\}$$

be the standard genetic code map (a many-to-one map onto 20 amino acids plus a stop signal); see, e.g., [16].

Fix any injective two-bit encoding $\mathsf{enc} : \{A, C, G, U/T\} \to \{0,1\}^2$ and extend it to a map

$$\mathsf{code} : \mathsf{Codon} \to \Omega_6$$

by concatenating the three two-bit blocks. Composing with the folding map yields a canonical $64 \to 21$ compression

$$\mathsf{Fold}_6^{\mathrm{bin}} \circ \mathsf{code} : \mathsf{Codon} \twoheadrightarrow X_6.$$

Remark 9.1 (Encoding search space is finite)**.** *If enc is required to be a bijection between the four nucleotides and $\{0,1\}^2$, then there are exactly $4! = 24$ possible encodings. Hence any optimization over encodings can be performed exhaustively (no sampling is needed).*

Testable comparison invariants. One can compare Gen and $\mathsf{Fold}_6^{\mathrm{bin}} \circ \mathsf{code}$ by encoding-independent statistics:

- the preimage-size histogram (degeneracy distribution);
- conditional entropy and mutual information for the induced random variables under a chosen codon prior;
- degeneracy conditioned on Hamming weight of the folded type in X_6;
- distances between degeneracy histograms (e.g. KL divergence after smoothing, or Earth Mover's Distance).

Remark 9.2 (What depends on enc)**.** *Because* code *is a bijection, the multiset of preimage sizes of* $\mathsf{Fold}_6^{\mathrm{bin}} \circ \mathsf{code}$ *equals that of* Fold_6 *(equivalently* $\mathsf{Fold}_6^{\mathrm{bin}}$*) and is therefore* independent *of* enc*. Encoding dependence enters through the* joint *distribution of the pair of outputs* $(\mathsf{Gen}(C), (\mathsf{Fold}_6^{\mathrm{bin}} \circ \mathsf{code})(C))$ *for a random codon* C*. A concrete, encoding-dependent objective is the mutual information*

$$I\big(\mathsf{Gen}(C);\ (\mathsf{Fold}_6^{\mathrm{bin}} \circ \mathsf{code})(C)\big),$$

under a specified codon prior (e.g. uniform).

The "three boundary states" coincidence. The decomposition $X_6 = X_6^{\mathrm{cyc}} \sqcup X_6^{\mathrm{bdry}}$ yields exactly three boundary states. Whether stop codons preferentially map into the boundary sector X_6^{bdry} is a concrete, encoding-dependent matching problem; no such alignment is assumed here.

Proposition 9.3 (What "stop $\subset$ boundary" means at window length 6)**.** *Let* $S_{\mathrm{stop}} \subset \mathsf{Codon}$ *be the set of stop codons, with* $|S_{\mathrm{stop}}| = 3$*, and fix a bijective two-bit encoding* enc *and the induced map* $\mathsf{code} : \mathsf{Codon} \to \Omega_6$*. Define the boundary-sector preimage set*

$$S_{\mathrm{bdry}}(\mathsf{enc}) := \big(\mathsf{Fold}_6^{\mathrm{bin}} \circ \mathsf{code}\big)^{-1}(X_6^{\mathrm{bdry}}) \subset \mathsf{Codon}.$$

Then $|S_{\mathrm{bdry}}(\mathsf{enc})| = 6$ *for every* enc*, hence* $S_{\mathrm{bdry}}(\mathsf{enc})$ *can never equal* S_{stop}*. Moreover, the inclusion* $S_{\mathrm{stop}} \subseteq S_{\mathrm{bdry}}(\mathsf{enc})$ *holds if and only if*

$$\mathrm{int}_6\big(\mathsf{code}(c)\big) \in \{14, 17, 19, 48, 51, 53\} \qquad \textit{for every } c \in S_{\mathrm{stop}}.$$

Proof. Proposition 4.3 gives $X_6^{\mathrm{bdry}} = \{100001, 100101, 101001\}$. By Theorem 6.15 each of these boundary words has exactly two preimages under Fold_6, namely

$$\mathrm{Fold}_6^{-1}(100001) = \{14, 48\}, \quad \mathrm{Fold}_6^{-1}(101001) = \{17, 51\}, \quad \mathrm{Fold}_6^{-1}(100101) = \{19, 53\},$$

so $|\mathrm{Fold}_6^{-1}(X_6^{\mathrm{bdry}})| = 6$. Since code is a bijection, $|S_{\mathrm{bdry}}(\mathsf{enc})| = 6$ as well. The membership condition is exactly the statement that stop codons land in the boundary-sector preimage under the index identification int_6. □

9.2 A three-factor template: a categorical interface problem

The standard model gauge group exhibits a three-factor product structure $SU(3)\times SU(2)\times U(1)$; see, e.g., [17]. Independently, the present framework isolates three stability channels φ–π–e. We stress that this mention is only motivational: no physical identification is assumed or used. What follows is a purely mathematical interface problem for future work.

Problem 9.4 (A hierarchical limit object with three natural projections (interface))**.** *There exists a hierarchy of finite-resolution objects generated by Hilbert refinement (spatial recursion) and Zeckendorf window growth (syntactic recursion), whose inductive limit defines a moduli object* $\mathcal{M}_\infty$ *together with three natural projection functors (or factor maps)*

$$\mathcal{M}_\infty \to \mathcal{M}_\varphi, \qquad \mathcal{M}_\infty \to \mathcal{M}_\pi, \qquad \mathcal{M}_\infty \to \mathcal{M}_{\mathrm{e}},$$

such that the associated linearized readout admits a three-sector decomposition compatible with these projections.

Remark 9.5 (What would make the interface nontrivial)**.** *In the minimal window-*6 *model, the* φ*- and* e*-constraints coincide (Remark after Proposition 5.9). A nontrivial realization of Problem 9.4 therefore requires a higher-resolution regime in which analytic stability (zeta/Abel holomorphy domains) and syntactic legality are no longer equivalent. Potential mathematical routes include weighted transfer operators, countable-state grammars, and zeta functions beyond the rational finite-state class.*

10 Conclusion

We presented a self-contained finite-resolution model of *resolution folding* on a 64-state local readout space. The central mathematical outputs are:

- the φ-stable sector of $\mathcal{H}_6 = \ell^2(\{0,1\}^6)$ has dimension 21 (the golden mean admissible words);
- the π-channel provides a canonical internal split $21 = 18 \oplus 3$ into cyclically admissible states and boundary states;
- the e-channel is expressed by the Artin–Mazur determinant identity $\zeta(z) = 1/\det(I - zA) = 1/(1 - z - z^2)$ and a spectral (Abel-type) normalization that places the dominant pole at the unit-circle boundary in the normalized variable;
- a computable surjection $\mathrm{Fold}_6 : \{0, \dots, 63\} \twoheadrightarrow X_6$ realizes a concrete $64 \to 21$ folding via Zeckendorf normalization and window truncation, with explicit preimage sets and degeneracy histogram (Theorem 6.15; Appendix E);
- beyond the base case, Fold_m admits a general preimage characterization and quantitative degeneracy control, including explicit bounds and observable scaling of the mean degeneracy (Section 6).

The model intentionally separates what is proven at the minimal window length from what becomes structurally nontrivial only at higher resolution: at window length 6, φ-legality and e-stability coincide as local constraints, while the π-channel refines the stable sector. Higher-resolution generalizations (longer windows, weighted transfer operators, and infinite-state extensions) provide a natural arena where the three channels may become genuinely independent. On the geometric side, the dihedral layout family acts by exact symmetries on Hilbert addressing, so adjacency-based statistics are rigid under layout changes (Section 2); on the syntactic side, the Zeckendorf recursion admits an explicit "uplift" decomposition at the level of languages and ℓ^2 spaces (Section 7).

Selected open directions. The remaining nontrivial questions are structural rather than computational:

- realize a genuinely natural multi-channel decomposition compatible with recursion and layout symmetries (Definition 8.3);

- formulate and analyze higher-resolution regimes where analytic stability is not equivalent to a single forbidden word, beyond finite-state rational zeta functions;
- treat external mapping templates (Section 9) as optimization/identifiability problems over finite encoding classes.

A Length-6 admissible words: the 21 elements of X_6 and the 18+3 split

A.1 The full admissible set X_6

The admissible set

$$X_6 = \{w \in \{0,1\}^6 : \ w \text{ contains no occurrence of } 11\}$$

has $|X_6| = 21$ elements (Corollary 3.2). Listed in lexicographic order:

000000	000001	000010	000100
000101	001000	001001	001010
010000	010001	010010	010100
010101	100000	100001	100010
100100	100101	101000	101001
101010			

A.2 Hamming-weight distribution

Let $|w|_1 := \sum_{i=1}^{6} w_i$ be the Hamming weight. The weight distribution of X_6 is

$$\#\{w \in X_6 : |w|_1 = k\} = \begin{cases} 1, & k = 0, \\ 6, & k = 1, \\ 10, & k = 2, \\ 4, & k = 3, \\ 0, & k \geq 4. \end{cases}$$

Indeed, choosing k ones with no adjacency is equivalent to choosing a k-subset of $\{1, \dots, 6\}$ with no consecutive integers, whose count is $\binom{6-k+1}{k}$.

A.3 Boundary states and cyclic states

The cyclic admissibility condition is the wrap-around constraint $w_6 w_1 \neq 11$. Thus the boundary set X_6^{bdry} consists of admissible words with $w_1 = w_6 = 1$:

$$X_6^{\mathrm{bdry}} = \{\texttt{100001},\ \texttt{100101},\ \texttt{101001}\}, \qquad |X_6^{\mathrm{bdry}}| = 3,$$

and the cyclic admissible set has size $|X_6^{\mathrm{cyc}}| = 18$ (Proposition 4.3).

B The layout group D_4 and a concrete 6-bit encoding of $D_4 \times D_4$

B.1 Group presentation and normal form

Let D_4 be the dihedral group of the square, presented as

$$D_4 = \langle r, s \mid r^4 = e,\ s^2 = e,\ srs = r^{-1} \rangle,$$

where r is a 90° rotation and s is a reflection across a fixed axis. Every element admits a unique normal form

$$g = r^k s^\epsilon, \qquad k \in \{0, 1, 2, 3\}, \quad \epsilon \in \{0, 1\}.$$

B.2 A three-bit encoding of D_4

Encode k in two bits and ϵ in one bit. Concretely, define

$$\mathrm{enc}_{D_4}(r^k s^\epsilon) := (b_1, b_2, b_3) \in \{0,1\}^3,$$

where (b_1, b_2) is the binary encoding of $k \in \{0,1,2,3\}$ and $b_3 = \epsilon$. This is a bijection $D_4 \to \{0,1\}^3$.

B.3 A 6-bit encoding of $D_4 \times D_4$

For a two-sector layout state $(g_+, g_-) \in D_4 \times D_4$, encode

$$\mathrm{enc}(g_+, g_-) := \mathrm{enc}_{D_4}(g_+) \parallel \mathrm{enc}_{D_4}(g_-) \ \in\ \{0,1\}^6,$$

where $\parallel$ denotes concatenation. Thus enc is a bijection $D_4 \times D_4 \to \{0,1\}^6$, giving an explicit choice of the identification ι in Section 2.4.

Remark B.1 (Non-canonicity and readout basis dependence)**.** *The encoding above is one of many valid identifications $D_4 \times D_4 \cong \{0,1\}^6$. Changing the encoding corresponds to a change of readout basis. In application interfaces (e.g. Section 9) this changes the preimage statistics of folding maps and should be treated as part of the model selection problem.*

C Zeckendorf computation and folding algorithms

C.1 Greedy Zeckendorf algorithm

Input: an integer $N \geq 0$. Output: Zeckendorf digits $(c_k)_{k\geq 1} \in \{0,1\}^{\mathbb{N}}$ such that $N = \sum_{k\geq 1} c_k F_{k+1}$ and $c_k c_{k+1} = 0$.

1. If $N = 0$, return $c_k = 0$ for all k.
2. Generate Fibonacci weights $(F_{k+1})_{k\geq 1}$ (with $F_1 = F_2 = 1$ and $F_{n+2} = F_{n+1} + F_n$) until $F_{K+1} \leq N < F_{K+2}$.
3. Initialize all digits $c_k := 0$.
4. For $k = K, K-1, \dots, 1$:
   - If $F_{k+1} \leq N$, set $c_k := 1$ and update $N \leftarrow N - F_{k+1}$.
   - If c_k was set to 1, skip the next index $k-1$ (to enforce $c_{k-1} = 0$).
5. Return (c_k).

Correctness and uniqueness are guaranteed by Theorem 6.2.

C.2 Computing Fold_6

Given an input integer $N \in \{0, \dots, 63\}$:

1. Compute $\mathrm{Z}(N) = (c_1, c_2, \dots)$ by the greedy algorithm above.
2. Output $\mathrm{Fold}_6(N) = (c_1, \dots, c_6) \in X_6$.

C.3 Local normalization viewpoint

Let a finite digit string $(d_1, \ldots, d_M) \in \{0,1\}^M$ represent an integer by Fibonacci weights, with digit d_k attached to weight F_{k+1}. Normalization to Zeckendorf form can be implemented via local rewrites induced by $F_n = F_{n-1} + F_{n-2}$ when digits are written in descending weight order; see, e.g., [8].

C.4 Reference implementation

Pure-Python (standard-library) reference implementations are included under the paper directory:

`docs/papers/2025_resolution_folding_phi_pi_e_hpa_omega/scripts/`.

D Zeta identities and normalization details

D.1 Factorization of $1 - z - z^2$

Let $\varphi = (1 + \sqrt{5})/2$. Then

$$1 - z - z^2 = (1 - \varphi z)(1 + \varphi^{-1} z).$$

Indeed,

$$(1 - \varphi z)(1 + \varphi^{-1} z) = 1 + (\varphi^{-1} - \varphi)z - z^2 = 1 - z - z^2,$$

since $\varphi - \varphi^{-1} = 1$. Thus the zeta function in Proposition 5.1 has poles at $z = \varphi^{-1}$ and $z = -\varphi$.

D.2 Unit-disk normalization

With the normalization $z = r/\varphi$ one has

$$\widetilde{\zeta}(r) = \zeta\Big(\frac{r}{\varphi}\Big) = \frac{1}{(1 - r)(1 + \varphi^{-2} r)}.$$

Hence $\widetilde{\zeta}$ is holomorphic on $|r| < 1$ and its principal pole sits at $r = 1$.

D.3 Comparison with the full shift

The full two-symbol shift has adjacency matrix

$$B = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, \qquad \rho(B) = 2, \qquad \det(I - zB) = 1 - 2z.$$

Thus its zeta function is $(1-2z)^{-1}$ with principal pole at $z = 1/2$. Under the same normalization $z = r/\varphi$, this pole becomes $r = \varphi/2 < 1$, i.e. it moves inside the unit disk. This cleanly illustrates the pole-barrier language: relative to the φ scaling, allowing the forbidden transition produces an interior pole.

E The complete Fold_6 table: outputs, Zeckendorf values, and preimages

This appendix records the full mapping data for $\mathrm{Fold}_6 : \{0, \ldots, 63\} \to X_6$. For each $w \in X_6$ we list:

- the word w in lexicographic order;

- its Zeckendorf value $V(w) = \sum_{k=1}^{6} w_k F_{k+1}$;
- the preimage $\mathrm{Fold}_6^{-1}(w) \subset \{0, \dots, 63\}$ (Theorem 6.15).

$w \in X_6$	$V(w)$	$\mathrm{Fold}_6^{-1}(w)$
000000	0	$\{0, 21, 34, 55\}$
000001	13	$\{13, 47\}$
000010	8	$\{8, 29, 42, 63\}$
000100	5	$\{5, 26, 39, 60\}$
000101	18	$\{18, 52\}$
001000	3	$\{3, 24, 37, 58\}$
001001	16	$\{16, 50\}$
001010	11	$\{11, 32, 45\}$
010000	2	$\{2, 23, 36, 57\}$
010001	15	$\{15, 49\}$
010010	10	$\{10, 31, 44\}$
010100	7	$\{7, 28, 41, 62\}$
010101	20	$\{20, 54\}$
100000	1	$\{1, 22, 35, 56\}$
100001	14	$\{14, 48\}$
100010	9	$\{9, 30, 43\}$
100100	6	$\{6, 27, 40, 61\}$
100101	19	$\{19, 53\}$
101000	4	$\{4, 25, 38, 59\}$
101001	17	$\{17, 51\}$
101010	12	$\{12, 33, 46\}$

F Fibonacci bounds and logarithmic cutoff estimates

This appendix records standard Fibonacci estimates used throughout the quantitative scaling arguments.

F.1 Binet formula and elementary bounds

Let $\varphi = \frac{1+\sqrt{5}}{2}$ and $\psi = \frac{1-\sqrt{5}}{2} = -\varphi^{-1}$. Then the Fibonacci numbers (F_n) with $F_1 = F_2 = 1$ satisfy the Binet formula

$$F_n = \frac{\varphi^n - \psi^n}{\sqrt{5}} \qquad (n \geq 1),$$

and in particular $F_n = \Theta(\varphi^n)$. Standard references include [15].

Remark F.1 (Relation to the main text)**.** *Lemma F.2 provides the two-sided bound used in Section 6. The main text records the same estimate as Lemma 6.1 together with Binet's formula.*

Lemma F.2 (Two-sided bounds)**.** *For all $n \geq 2$ one has*

$$\varphi^{n-2} \leq F_n \leq \varphi^{n-1}.$$

Proof. For $n = 2, 3$ this is immediate. Assume $n \geq 4$ and that the bounds hold for $n-1$ and $n-2$. Using $F_n = F_{n-1} + F_{n-2}$ and $\varphi^2 = \varphi + 1$ gives

$$F_n \leq \varphi^{n-2} + \varphi^{n-3} = \varphi^{n-3}(\varphi + 1) = \varphi^{n-1},$$

and similarly

$$F_n \geq \varphi^{n-3} + \varphi^{n-4} = \varphi^{n-4}(\varphi + 1) = \varphi^{n-2}.$$

□

F.2 Admissible density and the cutoff index

The admissible language size is $|X_m| = F_{m+2}$, hence the admissible density satisfies

$$\frac{|X_m|}{2^m} = \frac{F_{m+2}}{2^m} = \Theta\Big(\Big(\frac{\varphi}{2}\Big)^m\Big),$$

and the mean folding degeneracy is $\bar{d}_m = 2^m/F_{m+2} = \Theta((2/\varphi)^m)$.

The cutoff index $K(m)$ defined by $F_{K(m)+1} \le 2^m - 1 < F_{K(m)+2}$ satisfies $K(m) = m \log_\varphi 2 + O(1)$, as shown in Proposition 6.13.

G Reproducibility notes

This paper includes minimal, auditable scripts that reproduce the finite combinatorics and the folding-map statistics. All scripts are standard-library Python and live under:

`docs/papers/2025_resolution_folding_phi_pi_e_hpa_omega/scripts/`.

G.1 What is reproduced

The scripts reproduce:

- enumeration of X_6 and verification $|X_6| = 21$;
- Hamming-weight distribution of X_6;
- the $18 \oplus 3$ cyclic/boundary split;
- computation of $\mathrm{Fold}_6(N)$ for $N = 0, \ldots, 63$ and preimage-size statistics;
- generation of a complete Fold_6 table (Appendix E) listing $w \in X_6$, $V(w)$, and $\mathrm{Fold}_6^{-1}(w)$.
- generation of the Fold_m mean-degeneracy scaling rows used in Table 2.

G.2 Generated LaTeX fragments (optional)

If desired, scripts can write small LaTeX fragments (table rows) into

`docs/papers/2025_resolution_folding_phi_pi_e_hpa_omega/sections/generated/`

following the same pattern used elsewhere in the repository.

Scripts and fragments. In particular:

- `exp_fold6_stats.py` writes `fold6_degeneracy_rows.tex`. It also writes `fold6_full_table_rows.tex`.
- `exp_foldm_scaling_table.py` writes `foldm_scaling_rows.tex`.

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