The Geometry of Physical Constants in HPA–Ω: From the Fine-Structure Constant to Particle Spectra and Black-Hole/Cosmological Invariants

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Abstract

We formulate a geometricization program for physical constants based on Holographic Polar Arithmetic (HPA) and the Ω-framework. Observable constants are treated as geometric or spectral invariants of a scan–readout protocol, while the fixation of dimensional constants is treated as a metrological scale choice. We introduce a trichotomy of constants (defining constants, derived constants, and dimensionless invariants) and a general construction mapping geometric objects $\mathcal{G}(r)$ to constants $C(r)$ via $C(r) = F(I_r(\mathcal{G}))$, where r denotes a resolution/coarse-graining scale. Additive "impedance" variables are treated as logarithmic readout costs $C = -\log w$ of underlying multiplicative weights, so serial composition of constraints is multiplicative at the ontological level and additive only after the readout projection.

As an anchored worked example, we formulate a closed-theory geometric-impedance axiom for the electromagnetic coupling and identify three readout strata with phase spaces $U(1) \times SU(2)$, $SO(3)$, and $\mathbb{R}P^1$ arising from projective qubit kinematics (Hopf bundle and the $SU(2) \to SO(3)$ double cover). Using canonical volume normalizations, we derive the theorem-level value $\alpha_{\rm geo}^{-1} = 4\pi^3 + \pi^2 + \pi \approx 137.0363037759$. An exhaustive low-complexity integer search in the ansatz $a\pi^3 + b\pi^2 + c\pi$ shows that $(a, b, c) = (4, 1, 1)$ is the unique minimizer over the coefficient-sum domain $a, b, c \in \mathbb{Z}_{\geq 0}$ and $a + b + c \leq 10$.

We then extend the same volume quantization to a second worked example, the proton–electron mass ratio, deriving $\mu_{\rm geo} = m_p/m_e = 6\pi^5$, and to electroweak matching at the Z scale, deriving $\sin^2\theta_W = 3/13$ and $\alpha^{-1}(\mu_Z) = 13\pi^2$. We also incorporate QCD running and dimensional transmutation as part of the resolution-flow interface (recovering the $\overline{\rm MS}$ scale parameter in the PDG convention), and we record low-complexity rigidity signals for the CKM Jarlskog invariant and for the logarithmic Newton coupling at the proton scale. Finally, we derive the black-hole area law as the saturation of the covariant entropy bound expressed in boundary channel-counting form, and we list falsifiable intermediate claims together with an explicit error-budget interface.

Keywords: HPA; geometric impedance; fine-structure constant; readout geometry; spectral gap; Weinberg angle; Jarlskog invariant; black-hole entropy; scale flow.

Conventions. Unless otherwise stated, log denotes the natural logarithm. We keep a strict separation between (i) a metrology layer where defining constants are fixed by convention, (ii) a derived layer where dimensional constants inherit uncertainty through interfaces, and (iii) a dimensionless-invariant layer where geometricization targets live.

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Contents

1 Introduction: why "constants" admit geometricization

One of the central inputs of modern physics is a table of "constants": coupling strengths, mass spectra, mixing angles, and cosmological parameters. Empirically they are determined to high precision, yet a reusable template for why these numbers take their values, whether they obey structural relations, and whether such relations can be upgraded to geometric/algebraic invariants remains missing.

This paper takes a stance that reverses the usual order of presentation. Instead of starting from a Lagrangian and treating constants as coefficients, we start from *coordinate schematics and readout protocols*. The HPA viewpoint is that multiplicative structure (scaling, phase, periodicity) belongs to the ontological layer, whereas linear addition appears only at the readout layer as a projection–encoding operation. Consequently, physical constants are not treated as parameters in dynamical equations, but as *irreducible costs of readout* or *irreducible obstructions in spectral data* associated with a scan–readout protocol.

In the HPA–Ω pipeline, a conserved "whole" is accessed by a genuine unitary scan that introduces time as iteration count, and then mapped to discrete outcomes by an orthogonal-cut readout projection. The residual mismatch induced by discretization—the "gap" between multiplicative ontology and additive readout—enters the observable layer as an effective impedance and error budget.

Accordingly, whenever this paper uses an additive "impedance" or "cost" variable, it is a logarithmic readout of an underlying multiplicative weight (Section 3, Proposition 3.3). This is explicit for the inverse coupling α^{-1} and for the logarithmic Newton coupling $I_G = \log(\alpha_G^{-1})$.

Our goal is to formulate a publishable, reviewable structure with explicit separation of layers and with worked examples that fix the quantitative chain:

- which quantities are geometricization targets;
- how the geometric objects and invariants are defined;
- how metrological interfaces connect invariants to measured tables;
- which statements are imported external inputs, which are axioms, and which are derived theorems;
- a list of falsifiable intermediate claims.

1.1 Logic audit: external inputs, axioms, derived results, and fits

To keep the chain reviewable, we explicitly separate four categories: (i) *external inputs* (established metrology/QFT/GR results); (ii) *axioms* internal to the HPA–Ω constant-geometry program; (iii) *derived theorems* proven from these axioms (using standard mathematics as needed); (iv) *data-facing fits/rigidity checks* that test low-complexity ansätze against recommended values.

Complexity budgets for rigidity checks. Rigidity statements for integer/rational ansätze are meaningful only at bounded complexity: enlarging integer coefficients makes approximation progressively flexible. Accordingly, every rigidity proposition in this paper fixes an explicit finite search domain for its integer parameters, and uniqueness claims are relative to that domain.

2 A trichotomy of constants: metrology, derived quantities, and dimensionless invariants

2.1 Defining constants: fixation as a metrological scale choice

In the revised SI (2019), the unit system is anchored by fixing the numerical values of seven defining constants. In this setting, the question of "explaining the exact decimal values" of c, h, e, k_B, N_A, and $\Delta\nu_{\mathrm{Cs}}$ is not a well-posed physical prediction problem: their exact decimals encode a scale convention and interface by definition [1].

Remark 2.1 (Writing discipline)**.** *Defining constants are excluded from geometric prediction targets. They are treated as interfaces that map geometric invariants to SI readouts.*

2.2 Derived constants: interfaces plus dimensionless inputs

In the revised SI, the vacuum permeability μ_0 is no longer an exact defining quantity and becomes measurable; its uncertainty budget is controlled by the measurement of the dimensionless fine-structure constant α because c, h, e are fixed while α is not [2, 23].

2.3 Dimensionless invariants: where physics lives

Dimensionless quantities (e.g., α, mass ratios, mixing angles, $\Lambda\ell_P^2$) are invariant under unit changes and therefore serve as primary targets of geometricization. CODATA provides recommended values and uncertainties for α^{-1} [2], while the Particle Data Group (PDG) systematically summarizes Standard Model parameters and conventions [24].

3 Minimal inputs of the HPA–Ω framework

This section records the structural modules that will be repeatedly invoked. When a module is used later in the quantitative chain, we either prove it explicitly or cite a standard reference.

3.1 Multiplicative ontology and polar embedding

HPA takes the multiplicative monoid of positive integers $M = (\mathbb{N}_{>0}, \cdot)$ as a primitive structure and defines a polar embedding into $\mathbb{C}^*$ of the form

$$\mathcal{Z}(n) = \rho(n)\, \mathrm{e}^{\mathrm{i}\theta_\times(n)} \in \mathbb{C}^*, \tag{1}$$

where $\rho(n) > 0$ and $\theta_\times$ is additive under multiplication (mod 2π), so that $\mathcal{Z}$ is a multiplicative homomorphism.

3.2 A no-go principle: addition and multiplication cannot both be preserved

HPA uses a rigidity fact: except for the trivial real embedding, there is no embedding of $\mathbb{N}_{>0}$ into $\mathbb{C}^*$ that preserves both multiplication and linear addition. We use the following formulation.

Lemma 3.1 (Rigidity of additive–multiplicative embeddings on $\mathbb{N}_{>0}$)**.** *Let $f : \mathbb{N}_{>0} \to \mathbb{C}^*$ satisfy*

$$f(m+n) = f(m) + f(n), \qquad f(mn) = f(m)f(n), \tag{2}$$

for all $m, n \in \mathbb{N}_{>0}$. Then $f(n) = n$ for all $n \in \mathbb{N}_{>0}$.

Proof. By additivity, $f(n) = nf(1)$ for all n. By multiplicativity, $f(1) = f(1 \cdot 1) = f(1)^2$. Since $f(1) \in \mathbb{C}^*$, this implies $f(1) = 1$, hence $f(n) = n$. □

Therefore any nontrivial polar embedding with a genuine phase component cannot be additive, and "addition" can only enter through a readout projection/encoding. The resulting mismatch defines a structural gap that enters the observable layer.

3.3 Logarithmic readout: additive costs from multiplicative weights

The structural gap between multiplicative ontology and additive bookkeeping is resolved at the readout interface by a projection. In this paper, every additive "impedance" variable is a *log-cost*: a logarithmic image of an underlying multiplicative weight.

Definition 3.2 (Readout weight and log-cost). *Let $w \in (0,1]$ be a dimensionless readout weight (e.g., a success weight or likelihood of satisfying a constraint). Define the associated readout cost by*

$$C := -\log w. \tag{3}$$

Proposition 3.3 (Multiplicative-to-additive conversion under log readout). *If a layered readout protocol imposes serial constraints with weights $\{w_j\}_{j=1}^{J}$, so that the total weight is the product*

$$w_{\text{tot}} = \prod_{j=1}^{J} w_j, \tag{4}$$

then the total cost in Definition 3.2 is the sum of channel costs:

$$C_{\text{tot}} = -\log w_{\text{tot}} = \sum_{j=1}^{J} \big(-\log w_j \big) = \sum_{j=1}^{J} C_j. \tag{5}$$

Proof. This follows from $\log(\prod_j w_j) = \sum_j \log w_j$. □

Proposition 3.4 (Uniqueness of logarithmic costs under multiplicative composition). *Let $C : (0,1] \to [0,\infty)$ satisfy $C(1) = 0$ and the homomorphism rule*

$$C(xy) = C(x) + C(y) \tag{6}$$

for all $x, y \in (0,1]$. If C is continuous at 1 (equivalently, locally bounded on $(0,1]$), then there exists $k \geq 0$ such that

$$C(w) = -k \log w. \tag{7}$$

In particular, up to a choice of cost unit (the constant k), the logarithm is the unique additive readout of multiplicative weights.

Proof. Define $f : [0,\infty) \to [0,\infty)$ by $f(t) := C(\mathrm{e}^{-t})$. Then $f(t+s) = f(t) + f(s)$ and f is continuous at 0 by continuity of C at 1. The continuous Cauchy equation implies $f(t) = kt$ for some constant $k \geq 0$ [25]. Therefore $C(\mathrm{e}^{-t}) = kt$ and $C(w) = -k \log w$. □

Remark 3.5 (Log base and cost units). *Changing the logarithm base rescales C by a constant. Throughout we use the natural logarithm so that weights are written as $w = \exp(-C)$.*

3.4 Primitive projective phase spaces and closure of the candidate set

The worked examples in this paper use phase spaces that are fixed by projective quantum kinematics at minimal complexity. For a two-level interface, the ray space is $\mathbb{C}P^1 \cong S^2$ (Bloch sphere) and the associated principal $U(1)$ bundle of normalized state vectors is the Hopf fibration with total space $S^3 \cong SU(2)$ [26, 27]. Projectivization by $\{\pm 1\}$ gives the standard double cover $SU(2) \to SO(3)$ and $U(1) \to \mathbb{R}P^1$ [28].

Axiom 3.6 (Primitive projective phase-space closure). *In the minimal volume-quantized models of this paper, every internal phase space is built as a finite Cartesian product of primitives from the set*

$$\mathfrak{P} := \{U(1), SU(2), SO(3), \mathbb{R}P^1\}, \tag{8}$$

and finite disjoint unions thereof. All primitives are equipped with the standard unit-radius bi-invariant (or quotient) metrics, so the associated canonical volumes are fixed.

Proposition 3.7 (Minimal projective closure of the primitive set)**.** *Assume a primitive candidate set contains $U(1)$ and $SU(2)$ and is closed under the $\mathbb{Z}_2$ projectivization $M \mapsto M/\{\pm 1\}$. Then it contains*

$$SO(3) \cong SU(2)/\{\pm 1\}, \qquad \mathbb{R}P^1 \cong U(1)/\{\pm 1\}.$$

In particular, $\{U(1), SU(2), SO(3), \mathbb{R}P^1\}$ is the minimal such closure.

Proof. Closure under $M \mapsto M/\{\pm 1\}$ forces inclusion of the displayed quotients once $U(1)$ and $SU(2)$ are included, and minimality is immediate. □

Appendix G records the resulting canonical volumes and the $\mathbb{Z}_2$ quotient relations. This closure axiom is the source of the finite candidate sets used in the rigidity enumerations.

3.5 The scan operator Θ: time as iteration count

To decouple ontology (phase/multiplicative structure) from readout (additive projection), HPA introduces a genuine unitary scan operator Θ and defines time as the iteration count of Θ. The scan shift and the phase-multiplication operator form a Weyl pair [29, 30], producing a noncommutative structure and uncertainty-type tradeoffs.

Concretely, in standard quantum kinematics one models a Weyl pair by unitary operators U, V satisfying

$$UV = e^{i\omega} VU, \tag{9}$$

for some $\omega \in \mathbb{R}$. We encode the HPA scan–phase interface in the same form.

Axiom 3.8 (Scan–phase Weyl relation)**.** *There exist unitary operators Θ (scan shift) and Φ (phase multiplication) acting on the readout Hilbert space such that*

$$\Theta\Phi = e^{i\omega} \Phi\Theta \tag{10}$$

for some $\omega \in \mathbb{R}$ fixed by the scan slope/branch.

3.6 Golden branch and minimal binary readout

In the golden branch, the scan slope is selected such that the induced binary mechanical word becomes the Fibonacci word; the Ostrowski numeration degenerates to the Zeckendorf representation [31–33].

The golden slope is selected by the extremal Diophantine property of φ: its continued fraction has the minimal partial quotients, which makes rational approximations maximally difficult [20]. For Kronecker sequences $\{n\alpha\}$, discrepancy and gap statistics are controlled by continued-fraction data [21]. In particular, the three-distance theorem implies that, for any irrational α, the gaps between successive elements of $\{n\alpha\}$ take at most three values; the detailed "gap/step" structure is classical [22]. These external results justify treating the golden branch as a canonical low-complexity choice for binary readout localization.

Theorem 3.9 (Hurwitz bound and the golden extremizer)**.** *For any irrational α, there exist infinitely many rationals p/q such that*

$$\left| \alpha - \frac{p}{q} \right| < \frac{1}{\sqrt{5}\, q^2}. \tag{11}$$

The constant $1/\sqrt{5}$ is optimal in the sense that it cannot be replaced by a larger uniform constant over all irrational α, and the extremal case is attained (in the limit of best approximants) by $\alpha = \varphi$ and its $\mathrm{SL}(2, \mathbb{Z})$ transforms [20, 21].

Therefore the golden branch is a minimax choice for suppressing rational near-resonances in a finite-depth readout; this is the precise mathematical content of the phrase "most irrational".

4 General construction: constants as invariants of readout geometry

4.1 Geometric objects and invariants

We use a reusable geometricization template.

Definition 4.1 (A geometricization template for constants)**.** *Given a family of geometric objects $\mathcal{G}(r)$ that can depend on a resolution/coarse-graining scale r, define an invariant or spectral quantity*

$$I_r(\mathcal{G}) \in \mathbb{R}. \tag{12}$$

Define a physical constant (or a vector of constants) by

$$C(r) = F(I_r(\mathcal{G})), \tag{13}$$

where F is the mapping into the observational convention. The map F can contain metrological interfaces (defining constants) but must not introduce unit-dependent ambiguity into the dimensionless content.

Remark 4.2 (Data-facing matching inputs as multiplicative factors)**.** *When comparing a closed-theory value to a reference value, we parameterize the interface by multiplicative matching inputs. For a positive dimensionless invariant X, define*

$$s_X := \frac{X_{\text{ref}}}{X_{\text{geo}}}.$$

For an additive log-cost $C = -\log w$ (Definition 3.2), define the corresponding multiplicative matching input in the weight domain,

$$s_C := \frac{w_{\text{ref}}}{w_{\text{geo}}} = \exp\big(-(C_{\text{ref}} - C_{\text{geo}})\big).$$

These factors are the data-facing interface quantities in the multiplicative ontology; they are used explicitly for the electromagnetic weight (s_α), the proton–electron mass ratio (s_μ), and the proton Newton coupling (s_G).

In the HPA–Ω context, sources for $\mathcal{G}(r)$ include:

- topological channels induced by the readout cut (bulk/boundary/line);
- spectral data of defects in nonassociative sectors (e.g., octonionic or exceptional Jordan structures) [34–36];
- symbolic dynamical systems generated by scan–projection and their orbit statistics;
- one-dimensional compilation/routing cost fields (the Ω part) and the induced lapse.

4.2 Impedance as a prototype invariant: from metric length to process cost

In the HPA–Ω constant-geometry program, inverse couplings are identified with a *geometric impedance*: a minimal readout cost required to establish a stable interaction channel. At the ontological level, layered constraints multiply their readout weights, while the readout projection converts weights to additive costs by the log map (Proposition 3.3), which is unique up to a choice of cost unit under mild regularity (Proposition 3.4). Thus the ℓ^1-type accumulation of channel costs is the forced additive image of multiplicative composition rather than an independent postulate; this is instantiated for the electromagnetic worked example by Axiom 5.1.

5 An anchored worked example: a three-channel impedance model for α_{em}

5.1 Closed-theory axioms: serial composition and geometric impedance

We isolate the axioms that turn the three-term expression into a theorem-level statement inside the HPA–Ω constant-geometry program.

Axiom 5.1 (Serial composition: multiplicative constraints and logarithmic readout)**.** *If a readout protocol must satisfy channel constraints in a fixed hierarchy, assign each channel j a readout weight $w_j \in (0,1]$ and define its readout cost by $V_j := -\log w_j$. The total readout weight is the product*

$$w_{\text{tot}} = \prod_j w_j, \tag{14}$$

and the geometric impedance is the log-cost

$$\alpha_{\text{geo}}^{-1} := -\log w_{\text{tot}}. \tag{15}$$

Equivalently, by Proposition 3.3, the impedance is the serial sum

$$\alpha_{\text{geo}}^{-1} = \sum_j V_j. \tag{16}$$

Remark 5.2 (Numerical isolation of serial composition among standard aggregation rules)**.** *For the three-stratum costs $V_{\text{bulk}} = 4\pi^3$, $V_{\text{boundary}} = \pi^2$, and $V_{\text{line}} = \pi$, standard alternatives to serial addition give values far from the CODATA reference value. For example, the Euclidean aggregation $\sqrt{V_{\text{bulk}}^2 + V_{\text{boundary}}^2 + V_{\text{line}}^2}$ gives 124.4568430929 (relative deviation -9.18×10^{-2}), the max aggregation $\max\{V_{\text{bulk}}, V_{\text{boundary}}, V_{\text{line}}\}$ gives 124.0251067212 (relative deviation -9.49×10^{-2}), and the parallel-impedance aggregation $(V_{\text{bulk}}^{-1} + V_{\text{boundary}}^{-1} + V_{\text{line}}^{-1})^{-1}$ gives 2.3381204994 (relative deviation -9.83×10^{-1}). By contrast, the serial sum gives $4\pi^3 + \pi^2 + \pi = 137.0363037759$ (relative deviation $+2.22 \times 10^{-6}$).*

In the multiplicative domain of Axiom 5.1, define the total weight $w := \exp(-\alpha^{-1})$. Relative to the CODATA weight $w_{\text{CODATA}} = \exp(-\alpha_{\text{CODATA}}^{-1})$, the serial sum gives $w_{\text{geo}}/w_{\text{CODATA}} = \exp(-(\alpha_{\text{geo}}^{-1} - \alpha_{\text{CODATA}}^{-1})) \approx 9.99695 \times 10^{-1}$, whereas the Euclidean, max, and parallel rules give ratios 2.90×10^5, 4.47×10^5, and 3.15×10^{58} respectively.

Axiom 5.3 (Geometric impedance as normalized phase volume)**.** *For each channel j, the process cost V_j is identified with a normalized geometric invariant of the corresponding phase space $\mathcal{M}_j$. In the minimal model used here we take*

$$V_j = \text{Vol}(\mathcal{M}_j), \tag{17}$$

where Vol *is the canonical volume induced by the standard bi-invariant metric on compact Lie groups and its quotient metrics.*

5.2 Geometric identification of the three strata

The HPA readout is built from projective quantum kinematics: rays rather than vectors. For a minimal (two-level) readout interface, the ray space is $\mathbb{C}P^1 \cong S^2$ (Bloch sphere), and the associated principal $U(1)$ bundle of normalized state vectors is the Hopf fibration with total space $S^3 \cong SU(2)$ [26, 27]. The physical identification of ± 1 for spinorial lifts produces the standard double cover $SU(2) \to SO(3) = SU(2)/\{\pm 1\}$ [28].

Axiom 5.4 (Electromagnetic three-stratum phase spaces). *For the electromagnetic readout channel we take the three topologically distinguishable strata to be*

$$\mathcal{M}_{\text{bulk}} \cong U(1)\times SU(2), \qquad \mathcal{M}_{\text{boundary}} \cong SO(3) \cong \mathbb{R}P^3, \qquad \mathcal{M}_{\text{line}} \cong U(1)/\{\pm 1\} \cong \mathbb{R}P^1. \quad (18)$$

The quotient by $\{\pm 1\}$ *encodes the projective identification intrinsic to readout.*

Proposition 5.5 (Primitive factorization rigidity for the electromagnetic strata). *Fix the primitive set* $\mathfrak{P}$ *from Axiom 3.6 with canonical volumes (Appendix G). Impose the minimal factor counts compatible with the* π*-powers of the stratum costs: realize* $V_{\text{bulk}} = 4\pi^3$ *by a product of two primitives, and realize* $V_{\text{boundary}} = \pi^2$ *and* $V_{\text{line}} = \pi$ *each by a single primitive. Then the stratum phase spaces are forced (up to ordering) as*

$$\mathcal{M}_{\text{bulk}} \cong U(1)\times SU(2), \qquad \mathcal{M}_{\text{boundary}} \cong SO(3), \qquad \mathcal{M}_{\text{line}} \cong \mathbb{R}P^1.$$

Proof. From Appendix G one has $\text{Vol}(U(1)) = 2\pi$, $\text{Vol}(SU(2)) = 2\pi^2$, $\text{Vol}(SO(3)) = \pi^2$, and $\text{Vol}(\mathbb{R}P^1) = \pi$. Any two-factor product has the form $2^m\pi^k$ with $m \in \{0,1,2\}$ and $k \in \{2,3,4\}$. The constraint $k = 3$ forces exponent pattern $(1,2)$, and the coefficient 4 forces $m = 2$, hence the factors must be $U(1)$ and $SU(2)$.

For one-factor realizations, π^2 occurs only for $SO(3)$ and π occurs only for $\mathbb{R}P^1$ in the primitive set. □

5.3 A theorem-level value from group volumes

Theorem 5.6 (Three-channel impedance). *Under Axioms 5.1–5.4, the minimal-model geometric value of the inverse fine-structure constant is*

$$\alpha_{\text{geo}}^{-1} = 4\pi^3 + \pi^2 + \pi \approx 137.0363037759. \quad (19)$$

Proof. By Axiom 5.1, $\alpha_{\text{geo}}^{-1} = V_{\text{bulk}} + V_{\text{boundary}} + V_{\text{line}}$. By Axiom 5.3, $V_j = \text{Vol}(\mathcal{M}_j)$, and by Axiom 5.4 we reduce the calculation to canonical volumes. Appendix G records the standard values

$$\text{Vol}(U(1)) = 2\pi, \quad \text{Vol}(SU(2)) = 2\pi^2, \quad \text{Vol}(SO(3)) = \pi^2, \quad \text{Vol}(\mathbb{R}P^1) = \pi, \quad (20)$$

so that

$$V_{\text{bulk}} = \text{Vol}(U(1)\times SU(2)) = 4\pi^3, \qquad V_{\text{boundary}} = \pi^2, \qquad V_{\text{line}} = \pi. \quad (21)$$

Summing the three terms yields the stated expression. □

5.4 Quantitative rigidity: low-complexity integer relation search

To exclude coefficient-level tuning, we document a reproducible rigidity check: minimize the CODATA error within the three-term ansatz $a\pi^3 + b\pi^2 + c\pi$ over small nonnegative integers.

Let $\alpha_{\text{CODATA}}^{-1} = 137.035999177$ denote the CODATA 2022 recommended central value [2]. We exhaustively search all triples $(a,b,c) \in \mathbb{Z}_{\geq 0}^3$ with $a+b+c \leq 10$ and minimize the absolute error $|a\pi^3 + b\pi^2 + c\pi - \alpha_{\text{CODATA}}^{-1}|$. The coefficient-sum budget $a+b+c \leq 10$ is a fixed complexity constraint; it strictly contains the geometric triple $(4,1,1)$ (with sum 6) and tests robustness against nontrivial nearby coefficients at bounded total multiplicity.

Proposition 5.7 (Uniqueness at low coefficient complexity). *Within the coefficient-sum complexity domain* $a,b,c \in \mathbb{Z}_{\geq 0}$ *and* $a+b+c \leq 10$*, the unique minimizer of* $|a\pi^3+b\pi^2+c\pi-\alpha_{\text{CODATA}}^{-1}|$ *is* $(a,b,c) = (4,1,1)$*, with relative error* 2.22×10^{-6}*. The next-best triple in the same domain has relative error at least* 3.24×10^{-3}.

Proof. This is a finite check by exhaustive enumeration over all triples $(a, b, c) \in \mathbb{Z}^3_{\geq 0}$ with $a + b + c \leq 10$ and minimization of the absolute error. $\square$

Corollary 5.8 (Gap-robustness under target perturbations)**.** *Let $V = \{a\pi^3 + b\pi^2 + c\pi : \ a, b, c \in \mathbb{Z}_{\geq 0}, \ a + b + c \leq 10\}$ and let $T = \alpha^{-1}_{\text{CODATA}}$. Let $v_\star \in V$ be the unique minimizer and let $v_2 \in V \setminus \{v_\star\}$ be the best competitor. Define the margin*

$$m := |v_2 - T| - |v_\star - T| > 0.$$

Then for every perturbed target T' with $|T' - T| < m/2$, the minimizer over V remains uniquely $v_\star$. In particular, the large gap documented in Table 2 implies a wide robustness interval relative to metrological uncertainties.

Remark 5.9 (Interface to the period-realization viewpoint)**.** *The bounded-complexity search above is an instance of a general "selection under finite resources" signal: a large best-vs-second-best gap produces rigidity under perturbations. In the companion period-realization manuscript [37], this logic is abstracted as a gap-stability lemma for finite candidate classes and used as a quantitative ingredient in a selection principle.*

5.5 From α_{geo} to CODATA α: an error budget

The CODATA 2022 central value differs from the minimal geometric value by

$$\Delta\alpha^{-1} := \alpha^{-1}_{\text{CODATA}} - \alpha^{-1}_{\text{geo}} \approx -3.05 \times 10^{-4}. \tag{22}$$

In the multiplicative readout variable $w = \exp(-\alpha^{-1})$, this is the weight ratio

$$\frac{w_{\text{geo}}}{w_{\text{CODATA}}} = \exp(\Delta\alpha^{-1}) \approx 9.99695 \times 10^{-1}. \tag{23}$$

Equivalently, define the multiplicative matching input

$$s_\alpha := \frac{w_{\text{CODATA}}}{w_{\text{geo}}} = \exp(-\Delta\alpha^{-1}) \approx 1 + 3.05 \times 10^{-4}. \tag{24}$$

In the closed-theory reading, this gap is a controlled correction decomposed into (i) a renormalization map F between geometric impedance and operational coupling, and (ii) finite-resolution effects tied to the readout scale r.

In quantum field theory, effective electromagnetic couplings are scheme- and scale-dependent, with running controlled by a beta function. For example, one-loop QED running in a mass-independent scheme yields a logarithmic shift of α^{-1} with scale [3,4]. In our program, such scale dependence is absorbed into $r \mapsto \mu(r)$ and into the map F in Definition 4.1.

To quantify the size of the residual, treat $|\Delta\alpha^{-1}|$ as a scale-shift equivalent under one-loop QED running (31). At energies below the muon threshold, heavy charged fields decouple from the low-energy flow [39], so the electron contribution provides the relevant coefficient. For one Dirac fermion with $N_c = 1$ and $Q = 1$, the one-loop coefficient is $b = (2/3)N_c Q^2 = 2/3$ [4]. The scale ratio required to account for the offset satisfies

$$\log\left(\frac{\mu_{\text{eff}}}{\mu_0}\right) \approx \frac{2\pi}{b}\left(\alpha^{-1}_{\text{geo}} - \alpha^{-1}_{\text{CODATA}}\right) = 3\pi\left(\alpha^{-1}_{\text{geo}} - \alpha^{-1}_{\text{CODATA}}\right) \approx 2.87 \times 10^{-3}, \tag{25}$$

so $\mu_{\text{eff}}/\mu_0 \approx 1.0029$. Under the Fibonacci map (32), this corresponds to a sub-step correction $\delta r \approx 5.96 \times 10^{-3}$ of one unit of Zeckendorf depth. In this precise sense, the residual gap between α_{geo} and α_{CODATA} corresponds to a finite-resolution interface correction rather than a change of the coarse-graining scale by a full unit.

5.6 Metrological interfaces for electromagnetic constants (revised SI)

In the revised SI, c, h, e are defining constants while α remains experimentally determined; hence the uncertainty structure of μ_0 and ϵ_0 is controlled by that of α [1, 2, 23]. We record standard relations

$$\alpha = \frac{e^2}{4\pi\epsilon_0\hbar c} = \frac{\mu_0 e^2 c}{2h}, \qquad \epsilon_0 = \frac{1}{\mu_0 c^2}, \qquad Z_0 = \sqrt{\frac{\mu_0}{\epsilon_0}} = \mu_0 c. \tag{26}$$

A short derivation is provided in Appendix B.

6 Generalization I: running couplings as "resolution flow"

6.1 From energy scale μ to resolution r

In field-theoretic language, couplings run with the energy scale μ [3, 4]. In the HPA language, running originates from changes in the readout window, coding depth, and coarse-graining scale. We therefore introduce a map

$$\mu = \mu(r), \tag{27}$$

where r is instantiated as scan iteration depth, projection-window scale, Zeckendorf/Ostrowski bit depth, or effective compilation depth in the Ω framework, depending on context.

6.2 A concrete choice: Zeckendorf depth and a Fibonacci resolution map

To make the $r \mapsto \mu(r)$ interface quantitative, we fix the canonical golden-branch choice. Let r denote Zeckendorf depth: the maximal Fibonacci index used in the canonical decomposition. The Fibonacci numbers satisfy Binet's formula and grow exponentially, $F_r \sim \varphi^r/\sqrt{5}$ [40].

Lemma 6.1 (Zeckendorf depth and exponential growth)**.** *Let $n \in \mathbb{N}_{>0}$ have Zeckendorf representation with maximal Fibonacci index r, i.e. F_r appears and F_{r+1} does not. Then*

$$F_r \le n \le F_{r+1} - 1. \tag{28}$$

Consequently, $\log n = r \log\varphi + O(1)$ *as* $r \to \infty$*.*

Proof. The lower bound holds because the representation contains F_r. The upper bound is attained by the alternating sum

$$F_r + F_{r-2} + F_{r-4} + \cdots = F_{r+1} - 1, \tag{29}$$

which follows by induction using $F_{k+1} = F_k + F_{k-1}$. The asymptotic $\log n = r\log\varphi + O(1)$ follows from $F_r \sim \varphi^r/\sqrt{5}$ [40]. □

Axiom 6.2 (Fibonacci resolution–energy map)**.** *In the golden branch we identify the effective scale with the exponential growth of canonical depth and set*

$$\mu(r) = \mu_0\, \varphi^{\,r}, \tag{30}$$

where $\varphi = (1+\sqrt{5})/2$ and μ_0 is a conventional reference scale.

Equivalently, up to $O(1)$ factors, $\mu(r) \propto F_r$ by Binet asymptotics.

Appendix H records a data-facing calibration of this map by fixing $\mu_0 = m_e$ and listing the resulting $r(\mu)$ values at standard PDG reference scales, together with an inverse check that infers μ_0 from near-integer depth assignments.

6.3 A quantitative check against one-loop QED running

At one loop, the QED beta function gives a logarithmic running for α^{-1} with coefficient determined by charged matter content (in a mass-independent scheme and away from thresholds) [3,4]:

$$\alpha^{-1}(\mu) \;\approx\; \alpha^{-1}(\mu_0) \;-\; \frac{b}{2\pi}\log\!\left(\frac{\mu}{\mu_0}\right), \qquad b = \frac{2}{3}\sum_f N_c^{(f)} Q_f^2. \tag{31}$$

Combining (31) with Axiom 6.2 yields an r-linear flow:

$$\alpha^{-1}(r) \;\approx\; \alpha^{-1}(0) \;-\; \frac{b\log\varphi}{2\pi}\, r. \tag{32}$$

Using the CODATA 2022 low-energy value $\alpha^{-1}(0) = 137.035999177$ and the PDG effective value near the Z pole $\alpha^{-1}(m_Z) = 127.955$ [2,24], one finds $\Delta\alpha^{-1} = 9.080999$ over $\log(m_Z/m_e) = 12.0921$, which corresponds to an effective one-loop coefficient

$$b_{\text{eff}} := \frac{2\pi\,\Delta\alpha^{-1}}{\log(m_Z/m_e)} \approx 4.7186. \tag{33}$$

For comparison, heavy charged fields decouple from the low-energy flow [4, 39]. At $\mu = m_Z$ this excludes the top quark, so the active charged fermions are e, μ, τ and u, d, s, c, b. Their charge-weighted sum is

$$\sum_f N_c Q_f^2 = \underbrace{3}_{e,\mu,\tau}\ \underbrace{2\cdot 3\left(\frac{2}{3}\right)^2}_{u,c}\ \underbrace{3\cdot 3\left(\frac{1}{3}\right)^2}_{d,s,b} = \frac{20}{3}, \tag{34}$$

hence the one-loop coefficient is

$$b_{\text{SM}}(m_Z) = \frac{2}{3}\sum_f N_c Q_f^2 = \frac{40}{9} \approx 4.4444. \tag{35}$$

Numerically, $b_{\text{eff}}/b_{\text{SM}}(m_Z) \approx 1.062$. Threshold structure and hadronic-vacuum-polarization effects enter through matching [4, 24].

Within the HPA–Ω program, this paper isolates an explicit, falsifiable link between readout depth and scale flow at the leading-log level; threshold structure enters through finite matching corrections as in effective field theory [4, 24, 39].

6.4 Asymptotic freedom and dimensional transmutation in QCD

For QCD, the beta function has the opposite sign: the coupling decreases at high scales (asymptotic freedom) [4, 41, 42]. At one loop,

$$\frac{\mathrm{d}\alpha_s}{\mathrm{d}\log\mu} = -\frac{b_0}{2\pi}\,\alpha_s^2 + O(\alpha_s^3), \qquad b_0 = 11 - \frac{2}{3}n_f, \tag{36}$$

so that (away from thresholds)

$$\alpha_s(\mu) = \frac{2\pi}{b_0\log(\mu/\Lambda)}, \tag{37}$$

where Λ is the dimensional-transmutation scale (scheme-dependent). In the r-coordinate, a scheme change that rescales Λ by a constant factor corresponds to an additive shift of r (Proposition H.4).

At two loops, including $b_1 = 102 - \frac{38}{3}n_f$, the $\overline{\text{MS}}$ scale parameter is expressed as [4, 24]

$$\Lambda_{\overline{\text{MS}}} \;=\; \mu\,\exp\!\left(-\frac{2\pi}{b_0\,\alpha_s(\mu)}\right)\left(\frac{b_0\,\alpha_s(\mu)}{4\pi}\right)^{-\frac{b_1}{2b_0^2}}. \tag{38}$$

Using the PDG world average $\alpha_s(m_Z) = 0.1180 \pm 0.0009$ [24] with $n_f = 5$ gives $\Lambda^{(5)}_{\overline{\rm MS}} \approx 0.21\,\text{GeV}$.

With $n_f = 5$ one has $b_0 = 23/3$ and $b_1 = 116/3$. Taking $\mu = m_Z = 91.1876\,\text{GeV}$ and $\alpha_s(m_Z) = 0.1180$ in (38) yields $\Lambda^{(5)}_{\overline{\rm MS}} \approx 0.209\,\text{GeV}$. The PDG uncertainty ± 0.0009 yields the range $\Lambda^{(5)}_{\overline{\rm MS}} \approx (0.198\text{–}0.219)\,\text{GeV}$ [24].

Combining (37) with Axiom 6.2 yields an r-linear flow for α_s^{-1} analogous to (32), with slope $b_0 \log \varphi/(2\pi)$.

6.5 Electroweak matching: volume quantization and the Weinberg angle

We now record a closed-theory electroweak matching model at the Z-scale. The Standard Model relation between gauge couplings and the electromagnetic coupling is [4, 8–10]

$$\frac{1}{e^2} = \frac{1}{g^2} + \frac{1}{g'^2}, \qquad e = g \sin\theta_W = g' \cos\theta_W, \tag{39}$$

which implies (with $\alpha = e^2/(4\pi)$, $\alpha_2 = g^2/(4\pi)$, $\alpha_Y = g'^2/(4\pi)$)

$$\alpha^{-1} = \alpha_2^{-1} + \alpha_Y^{-1}, \qquad \sin^2\theta_W = \frac{\alpha}{\alpha_2} = \frac{\alpha_2^{-1}}{\alpha^{-1}}. \tag{40}$$

We adopt the following volume-quantization axiom at the electroweak matching scale.

Axiom 6.3 (Electroweak inverse couplings as weighted volumes)**.** *At a matching scale $\mu = \mu_Z$ (identified with the Z pole), the inverse electroweak couplings are identified with weighted canonical phase volumes:*

$$\alpha_2^{-1}(\mu_Z) = \dim(\mathfrak{su}(2))\,\mathrm{Vol}(SO(3)) = 3\pi^2, \tag{41}$$

and

$$\alpha_Y^{-1}(\mu_Z) = \left(\sum_{f\in \mathrm{SM}} Y_f^2\right) \mathrm{Vol}(SO(3)) = 10\pi^2, \tag{42}$$

where the sum is over the Standard Model chiral fermions in three generations, with hypercharges normalized by $Q = T_3 + Y$ [4, 24]. In the standard $SU(5)$ normalization $\alpha_1 = (5/3)\alpha_Y$, so $\alpha_1^{-1} = (3/5)\alpha_Y^{-1} = 6\pi^2$.

Explicitly, for one generation,

$$\sum Y^2 = 6\left(\frac{1}{6}\right)^2 + 3\left(\frac{2}{3}\right)^2 + 3\left(\frac{1}{3}\right)^2 + 2\left(\frac{1}{2}\right)^2 + 1\cdot 1^2 = \frac{10}{3}, \tag{43}$$

so for three generations $\sum_{f\in\mathrm{SM}} Y_f^2 = 10$.

Theorem 6.4 (Weinberg angle and $\alpha(\mu_Z)$ from volume quantization)**.** *Under Axiom 6.3 and the Standard Model relations* (40)*, one obtains*

$$\alpha^{-1}(\mu_Z) = 13\pi^2 \approx 128.3048572142, \qquad \sin^2\theta_W(\mu_Z) = \frac{3}{13} \approx 0.2307692308. \tag{44}$$

Proof. By Axiom 6.3, $\alpha_2^{-1}(\mu_Z) = 3\pi^2$ and $\alpha_Y^{-1}(\mu_Z) = 10\pi^2$. Using (40) gives $\alpha^{-1}(\mu_Z) = 13\pi^2$ and $\sin^2\theta_W(\mu_Z) = \alpha_2^{-1}/\alpha^{-1} = 3/13$. □

Numerical comparison. PDG quotes the effective electromagnetic coupling at the Z pole, $\alpha^{-1}_{\mathrm{PDG}}(\mu_Z) \approx 127.955$ [24]. It also quotes the $\overline{\mathrm{MS}}$ weak mixing angle, $\sin^2\theta_{W,\mathrm{PDG}}(\mu_Z) \approx 0.23122$ [24]. Theorem 6.4 then gives the explicit deviations

$$13\pi^2 - 127.955 \approx 3.50 \times 10^{-1} \qquad (2.73 \times 10^{-3} \text{ relative}), \tag{45}$$

and

$$\frac{3}{13} - 0.23122 \approx -4.51 \times 10^{-4} \qquad (-1.95 \times 10^{-3} \text{ relative}). \tag{46}$$

Equivalently, in multiplicative log-ratio form,

$$\log\left(\frac{13\pi^2}{\alpha^{-1}_{\mathrm{PDG}}(\mu_Z)}\right) \approx 2.73 \times 10^{-3}, \qquad \log\left(\frac{(3/13)}{\sin^2\theta_{W,\mathrm{PDG}}(\mu_Z)}\right) \approx -1.95 \times 10^{-3}. \tag{47}$$

Proposition 6.5 (Integer rigidity for $\alpha^{-1}(\mu_Z)$ in the ansatz $n\pi^2$)**.** *Let $\alpha^{-1}_{\mathrm{PDG}}(\mu_Z) \approx 127.955$ be the PDG reference value. Among all integers n with $1 \le n \le 50$, the unique minimizer of*

$$|n\pi^2 - \alpha^{-1}_{\mathrm{PDG}}(\mu_Z)| \tag{48}$$

is $n = 13$, i.e. $\alpha^{-1}(\mu_Z) \approx 13\pi^2$ is the unique best integer-π^2 approximation at this coefficient bound.

Proof. This is a finite check: evaluate $|n\pi^2 - \alpha^{-1}_{\mathrm{PDG}}(\mu_Z)|$ for $1 \le n \le 50$ and minimize. □

Remark 6.6 (Scale-shift equivalent under one-loop QED running)**.** *Treat $13\pi^2 - \alpha^{-1}_{\mathrm{PDG}}(\mu_Z)$ as a scale-shift equivalent under the one-loop QED running* (31) *with coefficient $b_{\mathrm{SM}} = 40/9$ at the electroweak scale. Then*

$$\log\left(\frac{\mu_Z}{\mu_*}\right) \approx \frac{2\pi}{b_{\mathrm{SM}}}\left(13\pi^2 - \alpha^{-1}_{\mathrm{PDG}}(\mu_Z)\right) \approx 4.94 \times 10^{-1}, \tag{49}$$

so $\mu_* \approx 0.61\,\mu_Z$. Under the Fibonacci map $\mu(r) = \mu_0\varphi^r$, this corresponds to a depth shift $\delta r \approx 1.03$. Since $\varphi^{-1} = 0.618\ldots$, the inferred scale ratio μ_*/μ_Z lies within a percent-level neighborhood of a single discrete Fibonacci step $\mu_Z \mapsto \mu_Z/\varphi$. Equivalently, one Fibonacci step induces a one-loop QED shift

$$\Delta\alpha^{-1}_{\varphi} := \frac{b_{\mathrm{SM}}}{2\pi}\log\varphi \approx 3.40 \times 10^{-1},$$

while the observed mismatch is $13\pi^2 - \alpha^{-1}_{\mathrm{PDG}}(\mu_Z) \approx 3.50 \times 10^{-1}$, differing by 9.6×10^{-3} (about 2.8% of $\Delta\alpha^{-1}_{\varphi}$).

Proposition 6.7 (Discrete rigidity of the electroweak integers)**.** *Let $\alpha^{-1}_{\mathrm{PDG}}(\mu_Z) \approx 127.955$ and $\sin^2\theta_{W,\mathrm{PDG}}(\mu_Z) \approx 0.23122$ be the PDG reference values [24]. Restrict to the π^2-quantized ansatz*

$$\alpha^{-1}(\mu_Z) = n\pi^2, \qquad \sin^2\theta_W(\mu_Z) = \frac{p}{n},$$

with an integer $1 \le n \le 50$ and a reduced rational p/n (equivalently, $\gcd(p,n) = 1$). Then the unique choice consistent with Propositions 6.5 and 6.8 is $(n,p) = (13,3)$. Consequently,

$$\alpha_2^{-1}(\mu_Z) = 3\pi^2, \qquad \alpha_Y^{-1}(\mu_Z) = 10\pi^2.$$

Proof. By Proposition 6.5, the unique best integer-π^2 approximation in the stated range is $n = 13$. With n fixed, Proposition 6.8 gives $p/n = 3/13$. The displayed values follow from $\alpha^{-1} = \alpha_2^{-1} + \alpha_Y^{-1}$ and $\sin^2\theta_W = \alpha_2^{-1}/\alpha^{-1}$. □

The same integers coincide with $\dim(\mathfrak{su}(2)) = 3$ and $\sum_{f\in\mathrm{SM}} Y_f^2 = 10$ under the Standard Model hypercharge assignments (Axiom 6.3), so the electroweak identification is simultaneously fixed by representation content and by low-complexity rigidity.

Proposition 6.8 (Rational rigidity at low denominator)**.** *Let $x \approx 0.23122$ be the PDG electroweak mixing reference value. Among all reduced rationals p/q with $1 \le q \le 50$ and $\gcd(p, q) = 1$, the closest to x is $3/13$.*

Proof. This is a finite check: enumerate all pairs (p, q) with $1 \le q \le 50$, $0 \le p \le q$, and $\gcd(p, q) = 1$, and minimize $|p/q - x|$. □

Proposition 6.9 (Geometric expression of running couplings)**.** *Given $\mathcal{G}(r)$ and an invariant $I_r(\mathcal{G})$, define an effective coupling by*

$$\alpha_{\mathrm{eff}}(\mu(r)) = F(I_r(\mathcal{G}))\,. \tag{50}$$

When increasing r corresponds to higher-resolution readout (shorter distances / higher energies), the running of α_{eff} is equivalent to the flow of I_r.

Proof. This is immediate: by definition, α_{eff} depends on scale only through the composite map $r \mapsto I_r(\mathcal{G}) \mapsto F(I_r)$, so varying r induces the running. □

This definition matches the PDG practice of quoting scale-dependent effective parameters and summarizing experimental constraints and theoretical frameworks [24].

6.6 A universal "multi-channel impedance" axiom

Extending the three-channel model for $\alpha_{\mathrm{em}}^{-1}$, we state the following structural axiom for general gauge couplings.

Axiom 6.10 (Multi-channel geometric impedance)**.** *For any effective coupling $\alpha_a(\mu)$, there exists a finite channel decomposition*

$$\alpha_{a,\mathrm{geo}}^{-1}(r) = \sum_{j=1}^{J_a} V_{a,j}(r), \tag{51}$$

where each $V_{a,j}$ corresponds to a topologically distinguishable sector in readout (e.g., bulk reservoir, boundary screen, line scan, or richer internal fiber submanifolds).

Axiom 6.11 (Volume quantization of channel costs)**.** *Each channel cost is an integer-weighted canonical phase volume built from the primitive set in Axiom 3.6: there exist products $\mathcal{M}_{a,j}$ of primitives and integers $n_{a,j}(r) \in \mathbb{N}_{>0}$ such that*

$$V_{a,j}(r) = n_{a,j}(r)\,\mathrm{Vol}(\mathcal{M}_{a,j}). \tag{52}$$

The scale dependence of $n_{a,j}(r)$ is piecewise constant in r, with discrete jumps only at thresholds, in the same sense as effective-field-theory decoupling and matching [4, 24, 39].

Together, Axioms 6.10 and 6.11 reduce "why is a coupling small/large?" to a finite rigidity problem: a bounded search over integer weights and a finite primitive phase-space set, plus threshold matching as part of the $r \mapsto \mu(r)$ interface.

7 Generalization II: mass ratios, spectral gaps, and mixing as holonomy

7.1 Mass as internal winding and extra scan cycles

Nonassociative structures (e.g., octonions and related nonassociative algebras) are standard mathematical objects in physics and geometry [34–36]. We adopt the scan-based mass–delay chain as an internal axiom of the program (Axiom D.1 in Appendix D), which interfaces the additional scan cycles required to resolve algebraic obstructions with effective time delays in the readout layer. To avoid unit dependence, we restrict the primary geometricization target to *dimensionless mass ratios.*

Definition 7.1 (Mass-ratio invariants)**.** *For particles i, j define*

$$\frac{m_i}{m_j} = \frac{I_i(\mathcal{G})}{I_j(\mathcal{G})}, \tag{53}$$

where $I_i(\mathcal{G})$ is a process cost or spectral quantity induced by the defect associated with particle i (e.g., associator tension, holonomy data, or a spectral gap).

This definition fixes ratios; an overall scale is set by the reference choice. The task reduces to defining a computable family $\{I_i\}$ whose ratios match the PDG mass-ratio data within uncertainties [24].

7.2 Worked example: proton–electron mass ratio

To make the program quantitative beyond α, we record a second low-parameter worked example: the proton–electron mass ratio $\mu = m_p/m_e$.

We strengthen Definition 7.1 by fixing a closed-theory invariant choice.

Axiom 7.2 (Mass invariants as phase-volume costs)**.** *For each particle class i, the mass invariant I_i is given by a canonical phase-volume of an internal compact manifold $\mathcal{M}_i$ (or a finite disjoint union of such manifolds), normalized so that the electron reference satisfies $I_e = 1$.*

$$\mathrm{Vol}\left(\bigsqcup_{a=1}^{A} \mathcal{M}_{i,a}\right) = \sum_{a=1}^{A} \mathrm{Vol}(\mathcal{M}_{i,a}). \tag{54}$$

For a baryon in three-color QCD ($N_c = 3$), the invariant is the sum over three color sectors [4, 24].

Remark 7.3 (Normalization gauge for mass invariants)**.** *Mass ratios depend only on the ratios I_i/I_j (Definition 7.1); the common multiplicative scale of $\{I_i\}$ is fixed by the reference choice $I_e = 1$ and plays the same role as a scale interface.*

Axiom 7.4 (Proton internal phase space)**.** *The proton is assigned three identical color sectors ($N_c = 3$) [4, 24]. Each color sector is modeled by the compact phase space*

$$\mathcal{M}_q \cong SO(3) \times SO(3) \times U(1), \tag{55}$$

so that $\mathcal{M}_p$ is the disjoint union of three copies of $\mathcal{M}_q$.

This choice is equivalent, at the level of canonical volumes, to the alternative factorization $SO(3) \times SU(2) \times \mathbb{R}P^1$, since $\mathrm{Vol}(SO(3))\,\mathrm{Vol}(U(1)) = \mathrm{Vol}(SU(2))\,\mathrm{Vol}(\mathbb{R}P^1)$ *by the $\mathbb{Z}_2$ quotient relations in Appendix G. We select $SO(3) \times SO(3) \times U(1)$ as the projective-rotation convention: rotational sectors are projectivized (spinorial double cover removed) while the phase circle is kept unquotiented.*

Theorem 7.5 (Proton–electron mass ratio)**.** *Under Axioms 7.2–7.4, the geometric prediction for the proton–electron mass ratio is*

$$\mu_{\text{geo}} := \frac{m_p}{m_e} = 6\pi^5 \approx 1836.1181087117. \tag{56}$$

Proof. By Axiom 7.2, $m_p/m_e = I_p/I_e = I_p$ since $I_e = 1$. By Axiom 7.4,

$$I_p = 3\,\mathrm{Vol}(SO(3) \times SO(3) \times U(1)). \tag{57}$$

Using $\mathrm{Vol}(SO(3)) = \pi^2$ and $\mathrm{Vol}(U(1)) = 2\pi$ (Appendix G) gives

$$I_p = 3\,(\pi^2)(\pi^2)(2\pi) = 6\pi^5. \tag{58}$$

□

Remark 7.6 (Minimal factor count for the π^5 scaling)**.** *Under the primitive set $\mathfrak{P}$ of Axiom 3.6 with canonical volumes (Appendix G), each primitive contributes at most a π^2 factor to a product volume. Therefore any product of two primitives has π-power at most 4, and the π^5 scaling of $\mu_{\text{geo}} = 6\pi^5$ forces at least three primitive factors per color sector. The three-factor search domain used in Proposition 7.7 is therefore the unique minimal domain that can realize the required scaling.*

Proposition 7.7 (Finite primitive factorization rigidity for μ)**.** *Fix the primitive candidate set $\{U(1), SU(2), SO(3), \mathbb{R}P^1\}$ from Axiom 3.6 with canonical volumes (Appendix G), and restrict a per-color sector $\mathcal{M}_q$ to be a product of three primitives (order irrelevant). Then the condition*

$$3\,\mathrm{Vol}(\mathcal{M}_q) = 6\pi^5 \tag{59}$$

holds if and only if

$$\mathcal{M}_q \cong SO(3) \times SO(3) \times U(1) \qquad \text{or} \qquad \mathcal{M}_q \cong SO(3) \times SU(2) \times \mathbb{R}P^1. \tag{60}$$

In particular, among the 20 *three-factor multisets, these two are the unique solutions; the next-closest candidate has relative deviation at least* 2.73×10^{-1} *(Table 3).*

Proof. Use $\mathrm{Vol}(U(1)) = 2\pi$, $\mathrm{Vol}(\mathbb{R}P^1) = \pi$, $\mathrm{Vol}(SO(3)) = \pi^2$, and $\mathrm{Vol}(SU(2)) = 2\pi^2$. Any three-factor product has the form

$$\mathrm{Vol}(\mathcal{M}_q) = 2^m \pi^k,$$

where m counts how many factors are drawn from $\{U(1), SU(2)\}$ and k is the sum of exponents 1 (for $U(1)$ or $\mathbb{R}P^1$) and 2 (for $SU(2)$ or $SO(3)$). The condition $3\,\mathrm{Vol}(\mathcal{M}_q) = 6\pi^5$ forces $m = 1$ and $k = 5$. Hence exactly two factors must contribute exponent 2 and one factor exponent 1.

If the exponent-1 factor is $U(1)$, then $m \geq 1$ already; to keep $m = 1$, both exponent-2 factors must be $SO(3)$, giving $SO(3) \times SO(3) \times U(1)$. If the exponent-1 factor is $\mathbb{R}P^1$, then m comes only from $SU(2)$ factors; to get $m = 1$ with two exponent-2 factors, exactly one is $SU(2)$ and the other is $SO(3)$, giving $SO(3) \times SU(2) \times \mathbb{R}P^1$. □

Numerically, CODATA 2022 gives $\mu_{\text{exp}} \approx 1836.15267343$ [2], so

$$\Delta\mu := \mu_{\text{geo}} - \mu_{\text{exp}} \approx -3.456 \times 10^{-2}, \qquad \frac{\Delta\mu}{\mu_{\text{exp}}} \approx -1.88 \times 10^{-5}. \tag{61}$$

Equivalently, in the multiplicative error metric,

$$\log\left(\frac{\mu_{\text{geo}}}{\mu_{\text{exp}}}\right) \approx -1.88 \times 10^{-5}. \tag{62}$$

Equivalently, define the multiplicative matching factor

$$s_\mu := \frac{\mu_{\rm exp}}{\mu_{\rm geo}} = \exp\left(-\log\frac{\mu_{\rm geo}}{\mu_{\rm exp}}\right) \approx 1 + 1.88 \times 10^{-5}. \tag{63}$$

In the closed-theory reading, s_μ is the matching input between the canonical phase-volume value and the CODATA ratio; it encodes QCD binding and radiative structure.

In QCD, the proton mass is dominated by dynamical scale generation and strong-interaction energy rather than by light-quark current masses [24, 43, 44]. Accordingly, the multiplicative matching input s_μ subsumes nonperturbative binding and radiative structure.

As a rigidity check analogous to Section 5, we minimize the CODATA error within the five-term ansatz

$$\mu(a, b, c, d, e) = a\pi^5 + b\pi^4 + c\pi^3 + d\pi^2 + e\pi \tag{64}$$

with small nonnegative integers.

Proposition 7.8 (Uniqueness at low coefficient complexity for μ)**.** *Within the coefficient-sum complexity domain $a, b, c, d, e \in \mathbb{Z}_{\geq 0}$ and $a + b + c + d + e \leq 10$, the unique minimizer of $|\mu(a, b, c, d, e) - \mu_{\rm exp}|$ is $(a, b, c, d, e) = (6, 0, 0, 0, 0)$, i.e., $\mu_{\rm geo} = 6\pi^5$, with relative error 1.88×10^{-5}. The next-best combination in the same domain has relative error at least 4.44×10^{-4}.*

Proof. This is a finite check by exhaustive enumeration over all $(a, b, c, d, e) \in \mathbb{Z}^5_{\geq 0}$ with $a + b + c + d + e \leq 10$. □

7.3 Mixing matrices as holonomy: from geometric phase to CKM/PMNS

The Ω framework uses standard geometric-phase language (Berry connection and holonomy) on parameter manifolds [5–7, 26]. We therefore adopt the following geometric implementation axiom.

Axiom 7.9 (Mixing matrices as holonomy)**.** *There exists a parameter manifold $\mathcal{M}$ associated with the readout protocol / gauge choice / internal fiber structure, together with a $U(1)$ or $U(n)$ connection $\mathcal{A}$ on $\mathcal{M}$, such that generation mixing can be represented as holonomy along basic loops:*

$$U_{\rm mix} = \mathcal{P}\exp\left(\oint_{\gamma\subset\mathcal{M}} \mathcal{A}\right). \tag{65}$$

Observable invariants (mixing angles, CP phases) correspond to conjugacy-class data of the holonomy.

This template upgrades "angles/phases" from fit parameters to geometric objects (connection, curvature, and topology). It matches the unitarity structure used in the standard CKM/PMNS parameterizations [45–48]. The PDG provides systematic reviews and conventions [24]. A short geometric-phase reminder is included in Appendix E.

7.4 A quantitative CP-odd reference: the Jarlskog invariant

For quark mixing, a basis-independent measure of CP violation is the Jarlskog invariant J, defined from the CKM matrix V by [49]

$$J := \operatorname{Im}(V_{ud}V_{cs}V^*_{us}V^*_{cd}), \tag{66}$$

equivalently by any of its rephasing-invariant forms. The PDG quotes [24]

$$J_{\rm PDG} = (3.00 \pm 0.15) \times 10^{-5}. \tag{67}$$

In the Wolfenstein parametrization [50], one has at leading order

$$J = A^2\lambda^6\bar{\eta} + O(\lambda^8), \tag{68}$$

with the standard $(\lambda, A, \bar{\rho}, \bar{\eta})$ conventions summarized by the PDG [24].

To complement the holonomy axiom with a minimal quantitative rigidity check, we test the one-parameter π-power ansatz

$$J_{\text{geo}}(a,n) = \frac{1}{a\pi^n}, \qquad a \in \mathbb{N}_{>0},\ n \in \mathbb{N}_{>0}, \tag{69}$$

which matches volume-normalized geometric costs and their inverses. Rigidity is stated at bounded multiplicity and exponent complexity; Propositions 7.10 and 7.11 fix the corresponding finite search domain.

Proposition 7.10 (Low-complexity π-power rigidity for CKM CP violation)**.** *Let* $J_{\text{PDG}} = 3.00 \times 10^{-5}$ *be the PDG central value. Among all pairs* (a,n) *with* $1 \le a \le 50$ *and* $1 \le n \le 20$*, the unique minimizer of* $|J_{\text{geo}}(a,n) - J_{\text{PDG}}|$ *is* $(a,n) = (11,7)$*, yielding*

$$J_{\text{geo}} = \frac{1}{11\pi^7} \approx 3.009942547 \times 10^{-5}, \tag{70}$$

with relative deviation 3.31×10^{-3}*, equivalently* $\log(J_{\text{geo}}/J_{\text{PDG}}) \approx 3.31 \times 10^{-3}$*. The next-best pair in the same domain has relative deviation at least* 9.37×10^{-3}*.*

Proof. This is a finite check by enumeration over (a,n) in the stated domain. □

Proposition 7.11 (Minimax rigidity over the PDG interval)**.** *Let the PDG reference interval be* $J \in [J_-, J_+]$ *with* $J_- := 2.85 \times 10^{-5}$ *and* $J_+ := 3.15 \times 10^{-5}$ *[24]. Among all pairs* (a,n) *with* $1 \le a \le 50$ *and* $1 \le n \le 20$*, the unique minimizer of the worst-case relative deviation*

$$\max\left\{\left|\frac{J_{\text{geo}}(a,n) - J_-}{J_-}\right|,\ \left|\frac{J_{\text{geo}}(a,n) - J_+}{J_+}\right|\right\} \tag{71}$$

is $(a,n) = (11,7)$*, with worst-case value* 5.61×10^{-2}*. The next-best pair in the same domain has worst-case value at least* 5.65×10^{-2}*.*

Proof. This is a finite check by enumeration over (a,n) in the stated domain, evaluating the worst-case error at the endpoints J_- and J_+. □

Closed-theory assignment. The rigidity winner $(a,n) = (11,7)$ is implemented by an explicit finite phase-space assignment.

Axiom 7.12 (CP-odd phase space and discrete multiplicity)**.** *Define the CP-odd readout sector by*

$$\mathcal{M}_{\text{CP}} \cong SO(3) \times SO(3) \times SO(3) \times \mathbb{R}P^1, \tag{72}$$

so $\text{Vol}(\mathcal{M}_{\text{CP}}) = \pi^7$ *under the canonical normalization (Appendix G). Define the discrete multiplicity by the combined gauge-sector dimension*

$$d_{\text{CP}} := \dim(\mathfrak{su}(3)) + \dim(\mathfrak{su}(2)) = 8 + 3 = 11, \tag{73}$$

and set

$$J_{\text{geo}} := \frac{1}{d_{\text{CP}}\,\text{Vol}(\mathcal{M}_{\text{CP}})} = \frac{1}{11\pi^7}. \tag{74}$$

This fixes both the exponent $n = 7$ and the coefficient $a = 11$ by geometry and finite group data rather than by fitting a free holonomy connection (Remark E.2).

Proposition 7.13 (Finite primitive factorization rigidity for the CP-odd phase space)**.** *Fix the primitive set $\mathfrak{P}$ from Axiom 3.6 with canonical volumes (Appendix G). Since each primitive contributes at most a π^2 factor, any product of three primitives has π-power at most 6, so the π^7 scaling forces at least four primitive factors. Among all products of exactly four primitives (order irrelevant), the condition*

$$\mathrm{Vol}(\mathcal{M}) = \pi^7 \tag{75}$$

holds if and only if

$$\mathcal{M} \cong SO(3) \times SO(3) \times SO(3) \times \mathbb{R}P^1.$$

Proof. From Appendix G, each primitive volume is one of $2\pi, 2\pi^2, \pi^2, \pi$. Any four-factor product has the form $2^m\pi^k$ with $m \in \{0,1,2,3,4\}$ and $k \in \{4,5,6,7,8\}$. The constraint $2^m\pi^k = \pi^7$ forces $m = 0$ and $k = 7$, so no factor can be $U(1)$ or $SU(2)$, and the exponent pattern must be $(2,2,2,1)$. Hence the factors are three copies of $SO(3)$ and one copy of $\mathbb{R}P^1$. □

8 Generalization III: c, gravity, and cosmology—from compilation lapse to dimensionless combinations

8.1 The status of c: defining constant and baseline propagation

In the SI, c belongs to the defining-constants layer [1]. In the HPA program, c is the baseline propagation speed in a regime where both gravitational stretching (delays) and high-energy "shortcuts" are negligible. We keep these two layers explicitly separated: metrologically, c is a fixed interface; physically, effective propagation is controlled by geometry and readout mechanisms.

8.2 Gravitational time delay as computational slowdown: lapse in the Ω framework

Define a computable "computational lapse" from one-dimensional nearest-neighbor compilation/routing costs. This provides an explicit interface with the standard relativistic lapse function that relates coordinate time to proper time in a $3+1$ decomposition [15].

Definition 8.1 (Computational lapse from routing costs)**.** *Let $\kappa(x)$ denote the local routing overhead and define*

$$\mathcal{N}(x) = \frac{\kappa_0}{\kappa(x)}, \qquad \mathrm{d}\tau_{\mathrm{loc}}(x) = \mathcal{N}(x)\,\mathrm{d}t. \tag{76}$$

Operationally: larger routing costs imply fewer intrinsic updates per unit background depth, corresponding to gravitational time dilation as computational slowdown.

Proposition 8.2 (Interface identification with a GR lapse)**.** *In a static spacetime with relativistic lapse $N(x)$ (so $\mathrm{d}\tau = N(x)\,\mathrm{d}t$ for static observers), define the routing-cost field by $\kappa(x) := \kappa_0/N(x)$. Then the computational lapse $\mathcal{N}(x)$ of Definition 8.1 equals the GR lapse $N(x)$. In Schwarzschild coordinates, $N(r) = \sqrt{1 - 2GM/(rc^2)}$, hence*

$$\frac{\kappa(r)}{\kappa_0} = \frac{1}{\sqrt{1 - 2GM/(rc^2)}} = 1 + \frac{GM}{rc^2} + O\!\left(\frac{G^2M^2}{r^2c^4}\right). \tag{77}$$

Proof. The first statement is immediate from the definitions. The Schwarzschild lapse is the standard g_{tt} relation [15], and the expansion uses $(1-x)^{-1/2} = 1 + x/2 + O(x^2)$. □

8.3 G and cosmological parameters: geometricize only dimensionless combinations

CODATA provides a recommended value for G with a comparatively large uncertainty [2]. We encode the interface discipline as an axiom.

Axiom 8.3 (Interface-only treatment of G)**.** *The Newton constant G is treated as a metrological interface constant that converts geometric/spectral costs into SI readouts. Consequently, the primary geometricization targets are dimensionless combinations such as $Gm^2/(\hbar c)$, $\Lambda\ell_P^2$, and $H_0 t_P$ rather than the decimal value of G itself.*

8.4 A worked example: gravitational fine-structure constant at the proton scale

Define the dimensionless Newton coupling (gravitational "fine-structure constant") for a particle of mass m by

$$\alpha_G(m) := \frac{Gm^2}{\hbar c}, \tag{78}$$

and in particular $\alpha_G(p) = Gm_p^2/(\hbar c)$ for the proton. The associated logarithmic inverse

$$I_G(m) := \log\big(\alpha_G(m)^{-1}\big) \tag{79}$$

is the scan–readout "impedance" variable: $\alpha_G(m) = \exp(-I_G(m))$, so I_G is the log-cost of the multiplicative weight α_G. This matches the role of logarithms in renormalization and dimensional transmutation [4, 24].

Axiom 8.4 (Gravitational three-stratum impedance in log form)**.** *At the proton scale, assign each stratum j a multiplicative weight $w_j^{(G)} := \exp(-V_j^{(G)})$ and define the geometric Newton coupling by*

$$\alpha_{G,\mathrm{geo}}(p) := \prod_j w_j^{(G)}. \tag{80}$$

The logarithmic impedance is the log-cost

$$I_G(p) := -\log \alpha_{G,\mathrm{geo}}(p) = V_{\mathrm{bulk}}^{(G)} + V_{\mathrm{boundary}}^{(G)} + V_{\mathrm{line}}^{(G)}, \tag{81}$$

where the costs $V_j^{(G)}$ are canonical phase volumes with phase spaces

$$\mathcal{M}_{\mathrm{bulk}}^{(G)} \cong U(1)\times SO(3), \qquad \mathcal{M}_{\mathrm{boundary}}^{(G)} \cong SU(2), \qquad \mathcal{M}_{\mathrm{line}}^{(G)} \cong U(1), \tag{82}$$

and costs $V_j^{(G)} = \mathrm{Vol}(\mathcal{M}_j^{(G)})$ under the canonical volume normalization.

Remark 8.5 (Primitive factorization equivalences at fixed volumes)**.** *Under the primitive set $\mathfrak{P} = \{U(1), SU(2), SO(3), \mathbb{R}P^1\}$ (Axiom 3.6), the canonical-volume identities in Appendix G imply the exchange*

$$\mathrm{Vol}(U(1)\times SO(3)) = \mathrm{Vol}(SU(2)\times \mathbb{R}P^1) = 2\pi^3, \qquad \mathrm{Vol}(SU(2)) = \mathrm{Vol}(U(1)\times \mathbb{R}P^1) = 2\pi^2.$$

Thus the bulk and boundary volumes in Axiom 8.4 admit equivalent primitive factorizations related by $\mathbb{Z}_2$ quotients; the choice records a projective convention, as in the proton phase-space assignment.

Proposition 8.6 (Primitive factorization rigidity for the gravitational strata)**.** *Fix the primitive set $\mathfrak{P}$ from Axiom 3.6 with canonical volumes (Appendix G). Impose the minimal factor counts compatible with the π-powers of the stratum costs: realize $V^{(G)}_{\text{line}} = 2\pi$ and $V^{(G)}_{\text{boundary}} = 2\pi^2$ each by a single primitive, and realize $V^{(G)}_{\text{bulk}} = 2\pi^3$ by a product of two primitives. Then*

$$\mathcal{M}^{(G)}_{\text{line}} \cong U(1), \qquad \mathcal{M}^{(G)}_{\text{boundary}} \cong SU(2),$$

and the bulk admits exactly the two equivalent factorizations

$$\mathcal{M}^{(G)}_{\text{bulk}} \cong U(1) \times SO(3) \qquad \text{or} \qquad \mathcal{M}^{(G)}_{\text{bulk}} \cong SU(2) \times \mathbb{R}P^1,$$

which have the same canonical volume (Remark 8.5).

Proof. By Appendix G, the primitive volumes are 2π, $2\pi^2$, π^2, π. For one-factor realizations, 2π occurs only for $U(1)$ and $2\pi^2$ occurs only for $SU(2)$ among primitives.

For two-factor products, the exponent condition π^3 forces exponent pattern $(1, 2)$ and the coefficient 2 forces exactly one factor to contribute a coefficient 2. Hence the two possibilities are $U(1) \times SO(3)$ and $SU(2) \times \mathbb{R}P^1$, which have equal volume by the displayed identities in Remark 8.5. □

Remark 8.7 (Numerical isolation of serial composition in the logarithmic variable)**.** *Let $V_1 := V^{(G)}_{\text{bulk}} = 2\pi^3$, $V_2 := V^{(G)}_{\text{boundary}} = 2\pi^2$, and $V_3 := V^{(G)}_{\text{line}} = 2\pi$. For the CODATA reference value $I_G(p) = \log(\alpha_G(p)^{-1}) \approx 88.024824446$, standard alternatives to serial addition give*

$$\begin{aligned}
\sqrt{V_1^2 + V_2^2 + V_3^2} &= 65.3809724311 \quad (\textit{rel.} \ -2.57 \times 10^{-1}),\\
\max\{V_1, V_2, V_3\} &= 62.0125533606 \quad (\textit{rel.} \ -2.96 \times 10^{-1}),\\
(V_1^{-1} + V_2^{-1} + V_3^{-1})^{-1} &= 4.4259282857 \quad (\textit{rel.} \ -9.50 \times 10^{-1}),
\end{aligned} \tag{83}$$

where "rel." denotes the relative deviation from $I_G(p)$. By contrast, the serial sum gives $V_1 + V_2 + V_3 = 2\pi^3 + 2\pi^2 + 2\pi = 88.0349474700$ (rel. $+1.15 \times 10^{-4}$).

In the multiplicative variable $\alpha_G = \exp(-I_G)$, these alternatives correspond to ratios $\alpha_G/\alpha_G(p)$ of order 6.82×10^9 (Euclidean), 1.98×10^{11} (max), and 2.03×10^{36} (parallel). The serial sum gives $\alpha_{G,\text{geo}}(p)/\alpha_G(p) \approx 9.90 \times 10^{-1}$.

Theorem 8.8 (Proton Newton coupling from canonical volumes)**.** *Under Axiom 8.4,*

$$I_G(p) = 2\pi^3 + 2\pi^2 + 2\pi \approx 88.034947470, \qquad \alpha_{G,\text{geo}}(p) = \exp(-I_G(p)) \approx 5.84666 \times 10^{-39}. \tag{84}$$

Proof. Using Appendix G, $\text{Vol}(U(1)) = 2\pi$, $\text{Vol}(SU(2)) = 2\pi^2$, and $\text{Vol}(SO(3)) = \pi^2$. Therefore

$$\text{Vol}(U(1) \times SO(3)) = (2\pi)(\pi^2) = 2\pi^3, \tag{85}$$

and the sum gives $I_G(p) = 2\pi^3 + 2\pi^2 + 2\pi$. Then $\alpha_{G,\text{geo}}(p) = \exp(-I_G(p))$ by the definition (79). □

Numerical comparison. CODATA 2022 gives $\alpha_G(p) = Gm_p^2/(\hbar c) \approx 5.90615 \times 10^{-39}$ [2], so the relative deviation of $\alpha_{G,\text{geo}}(p)$ is -1.01×10^{-2}. Equivalently,

$$\log\left(\frac{\alpha_{G,\text{geo}}(p)}{\alpha_G(p)}\right) \approx -1.01 \times 10^{-2}. \tag{86}$$

Equivalently, define the multiplicative matching input

$$s_G := \frac{\alpha_G(p)}{\alpha_{G,\text{geo}}(p)} = \exp\left(-\log\frac{\alpha_{G,\text{geo}}(p)}{\alpha_G(p)}\right) \approx 1 + 1.01 \times 10^{-2}. \tag{87}$$

Equivalently, in the logarithmic impedance variable, $I_G(p) = \log(\alpha_G(p)^{-1}) \approx 88.024824446$ (CODATA) versus $I_{G,\text{geo}}(p) = 2\pi^3 + 2\pi^2 + 2\pi \approx 88.034947470$, i.e. $\Delta I_G := I_{G,\text{geo}}(p) - I_G(p) \approx +1.01 \times 10^{-2}$ and $\Delta I_G/I_G(p) \approx +1.15 \times 10^{-4}$.

Proposition 8.9 (Low-complexity rigidity for $I_G(p)$)**.** *Let $I_G(p)$ be computed from CODATA via (79). Among all triples $(a, b, c) \in \mathbb{Z}^3_{\geq 0}$ with $a + b + c \leq 10$, the unique minimizer of*

$$\left|a\pi^3 + b\pi^2 + c\pi - I_G(p)\right| \tag{88}$$

is $(a, b, c) = (2, 2, 2)$. The next-best triple in the same coefficient-sum domain has relative error at least 4.94×10^{-3}.

Proof. This is a finite check: enumerate all triples $(a, b, c) \in \mathbb{Z}^3_{\geq 0}$ with $a + b + c \leq 10$ and minimize the absolute error. ☐

9 Black holes and boundaries: the area law as an extreme form of readout channel counting

Black-hole thermodynamics provides an established link between boundary geometry and entropy: the Bekenstein–Hawking entropy is proportional to horizon area [11, 13, 14, 16],

$$S_{\mathrm{BH}} = \frac{k_B c^3}{4G\hbar} A = \frac{k_B A}{4\ell_P^2}. \tag{89}$$

In the HPA language, readout is an orthogonal cut: it partitions a conserved whole into distinguishable channels and counts the residual mismatch as impedance. Accordingly, for horizons saturating the covariant entropy bound, the area law is the saturation of boundary readout channel count, and deviations are attributed to

1. lattice effects of discrete scan (e.g., non-Euclidean counting induced by canonical coding);
2. additional channel impedances from internal fiber defects;
3. finite-resolution coarse-graining in the readout window.

9.1 Channel counting and entropy bounds

We import the covariant entropy bound and the semiclassical area law as external inputs, and we isolate the minimal internal identification needed to interface scan–readout channel counting with gravitational entropy bounds.

Axiom 9.1 (Boundary readout entropy as channel count)**.** *For a boundary screen of area A at resolution r, let $\mathcal{N}_\partial(A, r)$ denote the maximal number of mutually distinguishable readout outcomes allowed by the protocol. Define the associated boundary readout entropy by*

$$S(A, r) := k_B \log \mathcal{N}_\partial(A, r). \tag{90}$$

This is the canonical channel-count entropy of a finite outcome set [51, 52].

Proposition 9.2 (Covariant entropy bound as a channel-capacity bound)**.** *The covariant entropy bound implies, at leading order,*

$$\log \mathcal{N}_\partial(A, r) \leq \frac{A}{4\ell_P^2}, \tag{91}$$

up to subleading corrections.

Proof. The covariant entropy bound gives $S(A, r) \leq k_B A/(4\ell_P^2)$ [19]. Substitute Axiom 9.1. ☐

Theorem 9.3 (Area law from boundary channel counting)**.** *If a horizon saturates the covariant entropy bound (as in the Bekenstein–Hawking entropy law), then*

$$S(A) = \frac{k_B A}{4\ell_P^2} \tag{92}$$

at leading order.

$$\mathcal{N}_\partial(A) = \exp\left(\frac{A}{4\ell_P^2}\right) \tag{93}$$

at the level of maximal channel count.

Proof. Saturation turns Proposition 9.2 into

$$\log \mathcal{N}_\partial(A) = \frac{A}{4\ell_P^2}.$$

Substitute Axiom 9.1. □

Remark 9.4 (Bekenstein bound route to the coefficient 1/4)**.** *The Bekenstein bound states $S \le 2\pi k_B ER/(\hbar c)$ for a system of energy E contained in radius R [12]. For a Schwarzschild black hole, take $E = Mc^2$, $R = R_s = 2GM/c^2$, and $A = 4\pi R_s^2$ [15]. Then*

$$S \le 2\pi k_B \frac{(Mc^2)(2GM/c^2)}{\hbar c} = \frac{4\pi k_B G M^2}{\hbar c} = k_B \frac{A}{4\ell_P^2}, \tag{94}$$

where $\ell_P^2 = G\hbar/c^3$. Thus the same coefficient 1/4 follows from the bound plus Schwarzschild geometry, with equality for black holes.

9.2 Subleading corrections as a data-facing target

Quantum and statistical-mechanical analyses of black-hole entropy generically predict subleading corrections to the area law, often of logarithmic form, with coefficients that depend on the field content and the ensemble [53–55].

$$S(A) = k_B \left(\frac{A}{4\ell_P^2} + \beta \log \frac{A}{\ell_P^2} + O(1)\right), \tag{95}$$

where β is theory/ensemble dependent [53–55].

Within the HPA–Ω program, these corrections are finite-resolution and discrete-coding effects in the boundary channel count $\mathcal{N}_\partial(A, r)$. Equivalently, they are controlled subleading terms in $\log \mathcal{N}_\partial(A, r)$ beyond the leading term $A/(4\ell_P^2)$.

Matching the sign and scaling of the leading correction provides a falsifiability interface for any explicit construction of $\mathcal{N}_\partial(A, r)$ from scan–readout primitives.

10 Falsifiable route and intermediate claims

To avoid a non-testable narrative, we isolate explicit falsifiable intermediate claims. Each item is refuted by numerical experiments or by data confrontation.

1. **Robustness of log-cost composition.** In the three-stratum α_{em} model, serial constraints multiply weights and the additive impedance is the logarithmic readout $-\log$ (Axiom 5.1, Propositions 3.3 and 3.4). Replacing the resulting serial sum by standard alternatives produces large deviations at fixed channel costs (Remark 5.2). The same isolation holds for the logarithmic Newton coupling $I_G(p)$ (Remark 8.7). This isolates the log-cost serial rule as the low-complexity composition compatible with the numeric reference values.

2. **Scale choice and RG/discreteness error budget.** The discrepancy $\alpha^{-1}_{\mathrm{CODATA}} - \alpha^{-1}_{\mathrm{geo}} \approx -3.05 \times 10^{-4}$ is mapped to a finite-resolution interface correction in Section 5: equivalently, the multiplicative matching input is $s_\alpha = w_{\mathrm{CODATA}}/w_{\mathrm{geo}} = \exp(-(\alpha^{-1}_{\mathrm{CODATA}} - \alpha^{-1}_{\mathrm{geo}})) \approx 1{+}3.05 \times 10^{-4}$. Under one-loop electron-only running the equivalent scale shift is $\mu_{\mathrm{eff}}/\mu_0 \approx 1.0029$, corresponding to a sub-step depth correction $\delta r \approx 5.96 \times 10^{-3}$ under the Fibonacci map.

3. **Resolution-map calibration.** With the golden-branch choice and the Fibonacci map $\mu(r) = \mu_0 \varphi^r$ (Axiom 6.2), the PDG reference scales $\{m_\mu, m_\tau, m_W, m_Z\}$ map to depths within $\delta_0 = 0.134$ of integers under the rigid reference choice $\mu_0 = m_e$ (Table 7). Under a uniform null for the fractional parts, the corresponding probability is $(2\delta_0)^4 \approx 5.2 \times 10^{-3}$ (Proposition H.2). In the same anchor set, the multiplicative minimax center satisfies $\mu_0^* = 0.998672\, m_e$ and yields $\delta_0^* = 0.131118$ (Proposition H.3).

4. **Electroweak matching rigidity.** The minimal volume-quantized prediction in Theorem 6.4 gives $\sin^2\theta_W(\mu_Z) = 3/13$ and $\alpha^{-1}(\mu_Z) = 13\pi^2$ with the explicit deviations recorded in Section 6. Proposition 6.5 isolates $n = 13$ as the unique best integer-π^2 approximation for $\alpha^{-1}(\mu_Z)$ at $1 \le n \le 50$, and Proposition 6.8 isolates 3/13 as the best reduced rational at denominator ≤ 50 for the PDG reference value. Proposition 6.7 combines the two to uniquely fix $\alpha_2^{-1} = 3\pi^2$ and $\alpha_Y^{-1} = 10\pi^2$.

5. **Dispersion signatures.** Discrete scan induces energy-dependent group-velocity corrections (Appendix F). Signal-front causality is enforced by analyticity at high frequency, which fixes $v_{\mathrm{front}} = c$ in causal linear response (Proposition F.2).

6. **Mass-ratio rigidity signals.** The proton–electron ratio is fixed by a finite phase-volume assignment (Theorem 7.5) together with a finite primitive factorization rigidity statement (Proposition 7.7, Table 3) and a low-complexity integer rigidity check (Proposition 7.8). The data-facing interface is a multiplicative matching input $s_\mu = \mu_{\mathrm{exp}}/\mu_{\mathrm{geo}} \approx 1 + 1.88 \times 10^{-5}$.

7. **CP-odd rigidity signals.** The CKM Jarlskog invariant exhibits a stable low-complexity π-power signature: Proposition 7.10 gives the unique best fit at the PDG central value, and Proposition 7.11 gives the unique minimax solution over the PDG interval in the same bounded search domain. The corresponding closed-theory realization is fixed by the CP-odd phase-space assignment and discrete multiplicity in Axiom 7.12, and the phase-space factorization is rigid at minimal factor count under the primitive set (Proposition 7.13).

8. **Gravitational coupling in log form.** The logarithmic Newton coupling $I_G(p) = \log(\alpha_G(p)^{-1})$ exhibits a stable low-complexity π-polynomial signature (Proposition 8.9) and the serial aggregation rule is isolated at fixed three-stratum costs (Remark 8.7). The phase-space assignment in Axiom 8.4 is tested directly against CODATA via Theorem 8.8, with multiplicative matching input $s_G = \alpha_G(p)/\alpha_{G,\mathrm{geo}}(p) \approx 1 + 1.01 \times 10^{-2}$. Its primitive factorizations are rigid at minimal factor count under the primitive set (Proposition 8.6).

11 Conclusion

We develop a geometricization framework for physical constants within HPA–Ω that is structured to be reviewable, extensible, and falsifiable.

- Conceptually, observable "constants" are framed as geometric/spectral invariants of a scan–readout protocol, rather than as coefficients written into equations from the start.

- Metrologically, we enforce a strict separation between defining constants (scale interfaces) and dimensionless invariants (physical targets).
- Methodologically, we provide a general template $C(r) = F(I_r(\mathcal{G}))$ together with explicit worked examples, including theorem-level values for α_{em}^{-1} and m_p/m_e and an electroweak matching prediction for $\sin^2\theta_W$. We also record a data-facing calibration of the Fibonacci resolution map against standard PDG scales.
- Physically, we formulate unified geometric languages for running couplings (as resolution flow, including QCD dimensional transmutation), mixing matrices and CP violation (as holonomy and rephasing invariants), gravitational time dilation (as computational lapse), and the black-hole area law (as saturation of an entropy bound expressed by boundary channel counting).
- Scientifically, we separate imported external inputs, internal axioms, derived theorems, and data-facing rigidity checks, and we list intermediate falsifiable claims.

Type	Statement / role in the chain
External	Revised SI as a defining-constants interface; CODATA recommended values as data-facing targets [1, 2].
External	Running couplings in QFT (scale dependence) and geometric phase/holonomy in quantum mechanics [3–7].
External	Electroweak relations among g, g', e and θ_W [4, 8–10].
External	Black-hole area law, entropy bounds, and the holographic principle/covariant entropy bound [11–19].
External	Continued-fraction Diophantine approximation and discrepancy control underlying the golden branch [20–22].
Axiom	Geometricization template $C(r) = F(I_r(\mathcal{G}))$ together with a logarithmic readout discipline and explicit phase-space/volume axioms for the worked examples (Sections 3, 5, 7, 6, 8, 9).
Derived	Minimal projective closure of the primitive candidate set (Proposition 3.7) and canonical group-volume identities for $U(1)$, $SU(2)$, $SO(3)$, and projective quotients (Appendix G).
Derived	Uniqueness of the logarithmic readout cost under multiplicative composition (Proposition 3.4).
Derived	Three-channel value $\alpha_{\text{geo}}^{-1} = 4\pi^3 + \pi^2 + \pi$ (Theorem 5.6).
Derived	Proton–electron mass ratio $\mu_{\text{geo}} = 6\pi^5$ (Theorem 7.5) and electroweak mixing $\sin^2\theta_W = 3/13$ (Theorem 6.4).
Derived	Proton Newton coupling in log form $I_G(p) = \log(\alpha_G(p)^{-1}) = 2\pi^3 + 2\pi^2 + 2\pi$ (Theorem 8.8).
Derived	Discrete-scan dispersion in a nearest-neighbor model (Proposition F.1) and interface identification of the computational lapse with a GR lapse (Proposition 8.2).
Derived	Area law from boundary channel counting under the covariant entropy bound plus saturation (Theorem 9.3).
Fit	Rigidity checks for low-complexity integer/rational ansätze (Propositions 5.7, 5.5, 7.7, 7.8, 6.5, 6.8, 6.7, 7.10, 7.11, 7.13, 8.6, and 8.9, Tables 2, 4, 3, 5, and 6; plus a data-facing calibration of the resolution map and near-integer depth tests (Propositions H.2, H.3, and H.4, Tables 7, 8, and 9).

Table 1: **Logic audit.** We mark what is imported from established literature, what is posited as an internal axiom of the HPA–Ω program, what is derived as a theorem, and what is tested as a low-complexity fit/rigidity check.

(a, b, c)	$a + b + c$	$a\pi^3 + b\pi^2 + c\pi$	Δ	$\Delta/\alpha_{\text{CODATA}}^{-1}$
$(4, 1, 1)$	6	137.0363037759	$+3.05 \times 10^{-4}$	$+2.22 \times 10^{-6}$
$(4, 0, 4)$	8	136.5914773356	-4.45×10^{-1}	-3.24×10^{-3}
$(3, 4, 1)$	8	135.6388402988	-1.40×10^{0}	-1.02×10^{-2}

Table 2: Exhaustive integer search in the ansatz $a\pi^3 + b\pi^2 + c\pi$ over the coefficient-sum domain $a, b, c \in \mathbb{Z}_{\geq 0}$ and $a + b + c \leq 10$. Here $\Delta = (a\pi^3 + b\pi^2 + c\pi) - \alpha_{\text{CODATA}}^{-1}$. The minimizer is $(4, 1, 1)$; the gap to the next best solution is large at fixed complexity budget. Integer-relation algorithms such as PSLQ provide a complementary, non-exhaustive route [38].

Per-color sector $\mathcal{M}_q$	$\mu_{\text{geo}} = 3\,\text{Vol}(\mathcal{M}_q)$	Δ	Δ/μ_{exp}
$SO(3) \times SO(3) \times U(1)$	$6\pi^5 = 1836.1181087117$	-3.46×10^{-2}	-1.88×10^{-5}
$SO(3) \times SU(2) \times \mathbb{R}P^1$	$6\pi^5 = 1836.1181087117$	-3.46×10^{-2}	-1.88×10^{-5}
$SU(2) \times U(1) \times U(1)$	$24\pi^4 = 2337.8181848161$	$+5.02 \times 10^{2}$	$+2.73 \times 10^{-1}$
$SO(3) \times U(1) \times U(1)$	$12\pi^4 = 1168.9090924080$	-6.67×10^{2}	-3.63×10^{-1}
$SU(2) \times U(1) \times \mathbb{R}P^1$	$12\pi^4 = 1168.9090924080$	-6.67×10^{2}	-3.63×10^{-1}
$SO(3) \times SO(3) \times \mathbb{R}P^1$	$3\pi^5 = 918.0590543558$	-9.18×10^{2}	-5.00×10^{-1}
$SO(3) \times SO(3) \times SO(3)$	$3\pi^6 = 2884.1675807259$	$+1.05 \times 10^{3}$	$+5.71 \times 10^{-1}$
$U(1) \times U(1) \times U(1)$	$24\pi^3 = 744.1506403272$	-1.09×10^{3}	-5.95×10^{-1}
$SO(3) \times U(1) \times \mathbb{R}P^1$	$6\pi^4 = 584.4545462040$	-1.25×10^{3}	-6.82×10^{-1}
$SU(2) \times \mathbb{R}P^1 \times \mathbb{R}P^1$	$6\pi^4 = 584.4545462040$	-1.25×10^{3}	-6.82×10^{-1}
$U(1) \times U(1) \times \mathbb{R}P^1$	$12\pi^3 = 372.0753201636$	-1.46×10^{3}	-7.97×10^{-1}
$SO(3) \times \mathbb{R}P^1 \times \mathbb{R}P^1$	$3\pi^4 = 292.2272731020$	-1.54×10^{3}	-8.41×10^{-1}
$U(1) \times \mathbb{R}P^1 \times \mathbb{R}P^1$	$6\pi^3 = 186.0376600818$	-1.65×10^{3}	-8.99×10^{-1}
$\mathbb{R}P^1 \times \mathbb{R}P^1 \times \mathbb{R}P^1$	$3\pi^3 = 93.0188300409$	-1.74×10^{3}	-9.49×10^{-1}
$SO(3) \times SU(2) \times U(1)$	$12\pi^5 = 3672.2362174234$	$+1.84 \times 10^{3}$	$+1.00 \times 10^{0}$
$SU(2) \times SU(2) \times \mathbb{R}P^1$	$12\pi^5 = 3672.2362174234$	$+1.84 \times 10^{3}$	$+1.00 \times 10^{0}$
$SO(3) \times SO(3) \times SU(2)$	$6\pi^6 = 5768.3351614518$	$+3.93 \times 10^{3}$	$+2.14 \times 10^{0}$
$SU(2) \times SU(2) \times U(1)$	$24\pi^5 = 7344.4724348468$	$+5.51 \times 10^{3}$	$+3.00 \times 10^{0}$
$SO(3) \times SU(2) \times SU(2)$	$12\pi^6 = 11536.6703229037$	$+9.70 \times 10^{3}$	$+5.28 \times 10^{0}$
$SU(2) \times SU(2) \times SU(2)$	$24\pi^6 = 23073.3406458073$	$+2.12 \times 10^{4}$	$+1.16 \times 10^{1}$

Table 3: Full enumeration over three-factor products of primitives $\{U(1), SU(2), SO(3), \mathbb{R}P^1\}$ (with canonical volumes) for a per-color sector $\mathcal{M}_q$. The only candidates achieving $\mu_{\text{geo}} = 3\,\text{Vol}(\mathcal{M}_q) = 6\pi^5$ are the two solutions in Proposition 7.7; the next candidate has order-one relative error. Here $\Delta = \mu_{\text{geo}} - \mu_{\text{exp}}$ with $\mu_{\text{exp}} \approx 1836.15267343$ [2].

(a,b,c,d,e)	$a{+}b{+}c{+}d{+}e$	$\mu(a,b,c,d,e)$	Δ	Δ/μ_{exp}
$(6,0,0,0,0)$	6	1836.1181087117	-3.46×10^{-2}	-1.88×10^{-5}
$(5,3,0,1,1)$	10	1835.3368940831	-8.16×10^{-1}	-4.44×10^{-4}
$(6,0,0,0,1)$	7	1839.2597013653	$+3.11 \times 10^{0}$	$+1.69 \times 10^{-3}$

Table 4: Rigidity check for $\mu = m_p/m_e$ in the ansatz $\mu(a,b,c,d,e) = a\pi^5 + b\pi^4 + c\pi^3 + d\pi^2 + e\pi$ over the coefficient-sum domain $a,b,c,d,e \in \mathbb{Z}_{\geq 0}$ and $a{+}b{+}c{+}d{+}e \leq 10$. Here $\Delta = \mu(a,b,c,d,e) - \mu_{\text{exp}}$. Exhaustive enumeration shows $(6,0,0,0,0)$ is the unique minimizer in the stated domain. Integer-relation algorithms such as PSLQ provide a complementary route [38].

a	n	$J_{\text{geo}}(a,n)$	Δ	Δ/J_{PDG}
11	7	$3.009942547 \times 10^{-5}$	$+9.94 \times 10^{-8}$	$+3.31 \times 10^{-3}$
35	6	$2.971889924 \times 10^{-5}$	-2.81×10^{-7}	-9.37×10^{-3}
34	6	$3.059298451 \times 10^{-5}$	$+5.93 \times 10^{-7}$	$+1.98 \times 10^{-2}$

Table 5: Rigidity check for the CKM Jarlskog invariant in the ansatz $J_{\text{geo}}(a,n) = 1/(a\pi^n)$ over $1 \leq a \leq 50$ and $1 \leq n \leq 20$. Here $\Delta = J_{\text{geo}}(a,n) - J_{\text{PDG}}$ with reference value $J_{\text{PDG}} = 3.00 \times 10^{-5}$ (PDG central value). Exhaustive enumeration shows $(a,n) = (11,7)$ is the unique minimizer in the stated domain.

(a,b,c)	$a{+}b{+}c$	$a\pi^3+b\pi^2+c\pi$	Δ	$\Delta/I_G(p)$
$(2,2,2)$	6	88.034947470	$+1.01\times10^{-2}$	$+1.15\times10^{-4}$
$(2,1,5)$	8	87.590121030	-4.35×10^{-1}	-4.94×10^{-3}
$(0,9,0)$	9	88.826439610	$+8.02\times10^{-1}$	$+9.11\times10^{-3}$

Table 6: Rigidity check for the proton logarithmic Newton coupling $I_G(p) = \log(\alpha_G(p)^{-1})$ in the ansatz $a\pi^3 + b\pi^2 + c\pi$ over the coefficient-sum domain $a,b,c \in \mathbb{Z}_{\geq 0}$ and $a+b+c \leq 10$. Here $\Delta = (a\pi^3 + b\pi^2 + c\pi) - I_G(p)$ with $I_G(p)$ computed from CODATA via (79). Exhaustive enumeration shows $(2,2,2)$ is the unique minimizer in the stated domain.

A Notation and layer conventions

We keep a strict separation of layers.

- **Ontology layer.** Multiplicative structure and phase structure are treated as primitives.
- **Scan layer.** A genuine unitary scan operator Θ introduces time as iteration count.
- **Readout layer.** A window projection and an orthogonal cut map continuous structure to discrete distinguishable outcomes; canonical coding (Ostrowski/Zeckendorf) fixes the readout grammar.
- **Observable layer.** The residual mismatch (gap/impedance) is the data-facing remainder and controls error budgets.

Resolution parameter r. The symbol r denotes a resolution/coarse-graining parameter. Depending on context it is instantiated as coding bit depth, window scale, scan depth, or (in the Ω setting) effective compilation depth. The only requirement is that there exists a map $r \mapsto \mu(r)$ connecting resolution flow to running couplings.

Complexity budgets. Finite rigidity checks are stated at bounded integer complexity. For nonnegative coefficient vectors $\mathbf{a} = (a_1, \dots, a_k) \in \mathbb{Z}_{\geq 0}^k$ in polynomial π-ansätze we use the coefficient-sum complexity

$$\|\mathbf{a}\|_1 := \sum_{i=1}^{k} a_i$$

and impose a fixed budget $\|\mathbf{a}\|_1 \leq B$ (e.g. $B = 10$ in the α^{-1}, $I_G(p)$, and μ integer searches). For rational approximations we bound the denominator $q \leq Q$ (e.g. $Q = 50$ for $\sin^2 \theta_W$). For primitive phase-space factorizations we bound the factor count. Every rigidity proposition states its finite domain explicitly.

Reproducibility. All finite rigidity checks and calibration tables in this paper are reproducible by a unified script included with the source: `scripts/hpa_omega_geometry.py`. Running `python3 scripts/hpa_omega_geometry.py emit-tex` writes the LaTeX row fragments into `sections/generated/`, which are input by Tables 2, 3, 4, 5, 6, and Appendix H. Running `python3 scripts/hpa_omega_geometry.py all` prints the corresponding values, multiplicative log-ratio error metrics, and search winners.

B Relations among electromagnetic constants (revised SI)

Starting from

$$\alpha = \frac{e^2}{4\pi\epsilon_0 \hbar c}, \tag{96}$$

and using

$$\epsilon_0 = \frac{1}{\mu_0 c^2}, \qquad \hbar = \frac{h}{2\pi}, \tag{97}$$

we obtain

$$\alpha = \frac{e^2}{4\pi\,(1/(\mu_0 c^2))\,(h/(2\pi))\,c} = \frac{\mu_0 e^2 c}{2h}. \tag{98}$$

Hence

$$\mu_0 = \frac{2\alpha h}{e^2 c}. \tag{99}$$

In the revised SI, c, h, e are defining constants while α is experimentally determined, so the uncertainty of μ_0 is controlled by that of α [1, 2, 23].

We also record

$$\epsilon_0 = \frac{1}{\mu_0 c^2}, \qquad Z_0 = \sqrt{\frac{\mu_0}{\epsilon_0}} = \mu_0 c. \tag{100}$$

C Substitutability list for the three-channel α model

Theorem 5.6 is a closed-theory statement conditional on explicit axiom choices. We list the main substitution points (each of which defines an alternative closed model that is tested against data).

1. **Channel-composition and readout projection.** Axiom 5.1 uses multiplicative composition of serial constraints ($w_{\text{tot}} = \prod_j w_j$) together with the logarithmic readout projection $\alpha_{\text{geo}}^{-1} = -\log w_{\text{tot}}$. Replacing either ingredient changes the value of α_{geo}^{-1} and is falsifiable by the same rigidity protocol.

2. **Impedance functional (volume vs alternative invariants).** Axiom 5.3 uses canonical volume. Replacing it by other geometric/spectral invariants (e.g., spectral determinants, analytic torsion, or normalized action functionals) yields alternative quantitative predictions.

3. **Normalization / scale.** The canonical-volume choice presupposes a unit-radius normalization for the underlying compact manifolds. Changing this normalization rescales the impedance and must be fixed by an independent calibration convention.

4. **Stratum identification.** Axiom 5.4 fixes the phase spaces as $U(1) \times SU(2)$, $SO(3)$, and $\mathbb{R}P^1$. Alternative identifications (e.g., different quotients or bundles) define distinct closed models.

D Mass–delay as a scan-based axiom

Axiom D.1 (Scan-based mass–delay chain)**.** *Internal algebraic obstructions (nonassociative defects) act as phase-impedance centers that require extra scan cycles to resolve locally [34–36]. The resulting increase in local scan density induces an effective metric stretching/time delay in the readout layer.*

The chain is

$$\begin{aligned} \text{nonassociative defect} &\longrightarrow \text{phase-impedance center} \\ &\longrightarrow \text{extra scan cycles} \\ &\longrightarrow \text{increased local scan density} \\ &\longrightarrow \text{metric stretching / time delay (Shapiro-like).} \end{aligned} \tag{101}$$

The relativistic dispersion $E^2 = p^2 + m^2$ provides a standard orthogonal decomposition between observable momentum and a rest-energy term. Empirically, gravitational time delay provides an operational reference for slowing effects [56].

External reference: Schwarzschild lapse and Shapiro delay. In Schwarzschild coordinates, the metric has $g_{tt} = -(1 - 2GM/(rc^2))c^2$, so a static clock at radius r satisfies

$$\mathrm{d}\tau = \sqrt{1 - \frac{2GM}{rc^2}}\,\mathrm{d}t, \tag{102}$$

exhibiting gravitational time dilation through the lapse factor $\sqrt{1 - 2GM/(rc^2)}$ [15].

Under the interface identification in Proposition 8.2, the corresponding routing-cost overhead is

$$\frac{\kappa(r)}{\kappa_0} = \left(1 - \frac{2GM}{rc^2}\right)^{-1/2} = 1 + \frac{GM}{rc^2} + O\left(\frac{G^2M^2}{r^2c^4}\right). \tag{103}$$

For light propagation past a gravitating body, the Shapiro time delay for a radar signal is, at leading post-Newtonian order,

$$\Delta t_{\text{Shapiro}} \approx \frac{2GM}{c^3} \log\left(\frac{4r_1r_2}{b^2}\right), \tag{104}$$

for endpoints at radii r_1, r_2 and impact parameter b [56, 57].

These standard formulas provide the external operational target for any scan-based identification of "extra cycles" with effective delays.

E Geometric-phase foundations for the holonomy template

For a family of normalized states $|\psi(\lambda)\rangle$ parameterized by λ on a manifold, define the Berry connection and curvature [5, 6, 26]

$$A = \mathrm{i}\,\langle\psi|\mathrm{d}\psi\rangle, \qquad F = \mathrm{d}A. \tag{105}$$

For a closed loop C, the geometric phase is

$$\gamma_{\text{geom}}[C] = \oint_C A = \int_{S:\,\partial S=C} F, \tag{106}$$

which is gauge-invariant at the level of curvature. The mixing-holonomy axiom in Axiom 7.9 uses the non-Abelian generalization with a $U(n)$ connection and path-ordered exponentiation [7, 26].

Proposition E.1 (Any unitary can be realized as holonomy on a circle)**.** *For any $U \in U(n)$, there exists a $U(n)$ connection on the trivial bundle over S^1 whose holonomy along the generator of $\pi_1(S^1)$ equals U.*

Proof. Choose a matrix logarithm H such that $U = \exp(\mathrm{i}H)$. On S^1 with angular coordinate $\theta \in [0, 2\pi]$, define a constant $u(n)$-valued connection one-form $\mathcal{A} = \mathrm{i}H\,\mathrm{d}\theta/(2\pi)$. Then the path-ordered exponential reduces to an ordinary exponential and

$$\mathcal{P}\exp\left(\oint_{S^1} \mathcal{A}\right) = \exp\left(\int_0^{2\pi} \frac{\mathrm{i}H}{2\pi}\,\mathrm{d}\theta\right) = \exp(\mathrm{i}H) = U.$$

Standard holonomy theory is developed in [58]. □

Remark E.2 (Predictivity requires geometric rigidity)**.** *Proposition E.1 shows that a holonomy representation is formally universal. Therefore, a predictive constant-geometry model must specify $\mathcal{M}$ and $\mathcal{A}$ by geometric rigidity (e.g., fixed bundles/metrics, minimal cycles, symmetry constraints), rather than by free fitting of a general connection.*

F On c and dispersion signatures: boundary conditions

In the "gravitational stretching vs high-energy shortcut" picture, the standard speed of light c arises as the baseline limit in which both effects are negligible.

Group velocity versus signal front. If a dispersive relation $\omega = \omega(k)$ is induced (e.g., by discrete scan), the phase and group velocities are

$$v_{\rm ph} = \frac{\omega}{k}, \qquad v_g = \frac{\mathrm{d}\omega}{\mathrm{d}k}. \tag{107}$$

It is standard that v_g can exceed c (or become negative) in dispersive media without enabling superluminal signaling; what constrains causality is the propagation of the wavefront, controlled by analyticity of the response function and the high-frequency limit [59, 60].

A minimal discrete-scan dispersion model. Discrete nearest-neighbor updates produce k-dependent group velocity. A standard example is the central-difference discretization of the one-dimensional wave equation.

Proposition F.1 (Dispersion from a nearest-neighbor discretized wave equation)**.** *Discretize $u_{tt} = c^2 u_{xx}$ on a uniform lattice $x_j = ja$ and times $t_n = n\Delta t$ by*

$$\frac{u_j^{n+1} - 2u_j^n + u_j^{n-1}}{\Delta t^2} = c^2 \frac{u_{j+1}^n - 2u_j^n + u_{j-1}^n}{a^2}. \tag{108}$$

Plane-wave solutions $u_j^n = \exp(\mathrm{i}(kja - \omega n\Delta t))$ satisfy

$$\sin^2\left(\frac{\omega\Delta t}{2}\right) = \nu^2 \sin^2\left(\frac{ka}{2}\right), \qquad \nu := \frac{c\Delta t}{a}, \tag{109}$$

and the group velocity is

$$v_g(k) = \frac{\mathrm{d}\omega}{\mathrm{d}k} = c\,\frac{\cos(ka/2)}{\sqrt{1 - \nu^2 \sin^2(ka/2)}}. \tag{110}$$

In particular, $\omega = ck + O(k^3)$ as $k \to 0$, while v_g depends on k for $\nu \neq 1$.

Proof. Substitute the plane-wave ansatz into the update equation and simplify using $\cos x = 1 - 2\sin^2(x/2)$. Differentiate the dispersion relation implicitly to obtain $v_g(k)$, and use $\sin x = x + O(x^3)$ for the low-k expansion. □

Proposition F.2 (Front velocity from analyticity)**.** *In causal linear response, the effective refractive index $n(\omega)$ is analytic in the upper half-plane and satisfies $n(\omega) \to 1$ as $\omega \to \infty$. Consequently the wavefront (signal-front) velocity equals c:*

$$v_{\rm front} = c. \tag{111}$$

Proof. Define

$$v_{\rm front} := \lim_{\omega\to\infty} \frac{\omega}{k(\omega)} = \frac{c}{\lim_{\omega\to\infty} n(\omega)}, \tag{112}$$

and the stated analyticity and high-frequency limit imply $\lim_{\omega\to\infty} n(\omega) = 1$ and hence $v_{\rm front} = c$ [59, 60]. □

Boundary condition for discrete-scan dispersion. Accordingly, any scan-induced dispersion is treated at the level of $v_g(\omega)$, while compatibility with locality/causality is enforced by Proposition F.2.

G Canonical volumes of the phase spaces

We record the elementary volume computations used in Theorem 5.6.

Lemma G.1 (Volumes of $U(1)$, $SU(2)$, $SO(3)$, and projective quotients)**.** *With the standard (unit-radius) normalizations,*

$$\mathrm{Vol}(U(1)) = 2\pi, \qquad \mathrm{Vol}(SU(2)) = \mathrm{Vol}(S^3) = 2\pi^2. \tag{113}$$

Moreover, the $\mathbb{Z}_2$ quotients satisfy

$$\mathrm{Vol}(SO(3)) = \mathrm{Vol}(SU(2)/\{\pm 1\}) = \pi^2, \qquad \mathrm{Vol}(\mathbb{R}P^1) = \mathrm{Vol}(U(1)/\{\pm 1\}) = \pi. \tag{114}$$

Consequently,

$$\mathrm{Vol}(U(1) \times SU(2)) = 4\pi^3. \tag{115}$$

Proof. The first identity is the circumference of the unit circle.

For $SU(2) \cong S^3$, use hyperspherical coordinates on S^3 with volume element

$$\mathrm{d}V = \sin^2\chi \, \sin\theta \, \mathrm{d}\chi \, \mathrm{d}\theta \, \mathrm{d}\phi, \qquad \chi \in [0,\pi], \ \theta \in [0,\pi], \ \phi \in [0, 2\pi]. \tag{116}$$

Then

$$\mathrm{Vol}(S^3) = \int_0^\pi \sin^2\chi \, \mathrm{d}\chi \int_0^\pi \sin\theta \, \mathrm{d}\theta \int_0^{2\pi} \mathrm{d}\phi = \left(\frac{\pi}{2}\right) \cdot 2 \cdot 2\pi = 2\pi^2. \tag{117}$$

The quotient maps $SU(2) \to SU(2)/\{\pm 1\} \cong SO(3)$ and $U(1) \to U(1)/\{\pm 1\} \cong \mathbb{R}P^1$ are two-sheeted Riemannian coverings under the induced quotient metrics. Let $\pi : M \to M/\{\pm 1\}$ denote either map and $\mathrm{d}V, \mathrm{d}\bar{V}$ the corresponding volume forms. Since π is a local isometry one has $\pi^*(\mathrm{d}\bar{V}) = \mathrm{d}V$, and since the covering has degree 2,

$$\mathrm{Vol}(M) = \int_M \mathrm{d}V = \int_M \pi^*(\mathrm{d}\bar{V}) = 2 \int_{M/\{\pm 1\}} \mathrm{d}\bar{V} = 2\,\mathrm{Vol}(M/\{\pm 1\}),$$

so the quotient volumes are halved.

Finally, $\mathrm{Vol}(U(1) \times SU(2)) = \mathrm{Vol}(U(1))\,\mathrm{Vol}(SU(2)) = (2\pi)(2\pi^2) = 4\pi^3$. □

H Calibration of the Fibonacci resolution map

This appendix records a data-facing calibration of the Fibonacci resolution–energy map in Axiom 6.2. We fix the reference scale to the electron mass,

$$\mu_0 = m_e, \tag{118}$$

and define

$$r(\mu) := \frac{\log(\mu/m_e)}{\log\varphi}. \tag{119}$$

With r identified as Zeckendorf depth (maximal Fibonacci index), integer shifts $r \mapsto r+1$ correspond to multiplicative rescaling by φ.

Table 7 places the muon and tau thresholds near $r \approx 11$ and $r \approx 17$, and the electroweak reference scale near $r \approx 25$. This anchors r as a discrete depth coordinate with finite matching corrections at thresholds, matching leading-log running combined with decoupling/matching in effective field theory [4, 24, 39].

The anchor set $\{m_\mu, m_\tau, m_W, m_Z\}$ consists of scheme-stable threshold scales used in the standard running-coupling conventions. By Proposition H.4, any multiplicative matching factor $\mu \mapsto s\mu$ induces an additive shift in r, so composite/hadronic scales enter as matching inputs rather than as integer-depth anchors.

Scale μ	μ [GeV]	$r(\mu)$	$r - \text{round}(r)$
m_e	5.1099895×10^{-4}	0.000	0.000
m_μ	1.0565838×10^{-1}	11.080	+0.080
m_τ	1.77686	16.945	−0.055
m_p	0.9382721	15.618	−0.382
m_W	80.377	24.866	−0.134
m_Z	91.1876	25.128	+0.128

Table 7: Calibration of the Fibonacci resolution coordinate $r(\mu) = \log(\mu/m_e)/\log\varphi$ at standard particle-physics scales (values from PDG conventions [24]). The near-integer depth tests in this appendix use the leptonic/electroweak subset $\{m_\mu, m_\tau, m_W, m_Z\}$; the proton scale is included for comparison and enters as a matching input rather than an integer-depth anchor.

Scale μ	$r_*(\mu)$	$\mu_0(\mu)$ [GeV]	$\mu_0(\mu)/m_e$	$\mu_0(\mu)/m_e - 1$
m_μ	11	5.3093×10^{-4}	1.0390	$+3.90 \times 10^{-2}$
m_τ	17	4.9758×10^{-4}	0.9737	-2.63×10^{-2}
m_W	25	4.7912×10^{-4}	0.9376	-6.24×10^{-2}
m_Z	25	5.4356×10^{-4}	1.0637	$+6.37 \times 10^{-2}$

Table 8: Inferred reference scales $\mu_0(\mu) = \mu/\varphi^{\text{round}(r(\mu))}$ from near-integer depth assignments for standard leptonic/electroweak scales.

Definition H.1 (Uniform null model for fractional parts)**.** *Fix reference scales* $\{\mu_i\}_{i=1}^N$ *and define* $\delta_i := r(\mu_i) - \text{round}(r(\mu_i)) \in [-1/2, 1/2]$*. The* uniform null model *is that the fractional parts of* $\{r(\mu_i)\}$ *are independent and uniform on* $[0,1]$ *(equidistribution theory provides the canonical uniform baseline) [21]. This is a statistical null model used to quantify how restrictive the near-integer event is; it is not an additional axiom of the HPA–Ω program.*

Proposition H.2 (Near-integer depth probability under the uniform null)**.** *Under Definition H.1,*

$$\mathbb{P}\left(\max_{1\le i\le N} |\delta_i| \le \delta_0\right) = (2\delta_0)^N. \tag{120}$$

For $\{\mu_i\} = \{m_\mu, m_\tau, m_W, m_Z\}$ *one has* $\delta_0 = 0.134$ *from Table 7 and* $N = 4$*, hence* $(2\delta_0)^N \approx 5.2 \times 10^{-3}$.

Proof. Under the null, each δ_i is uniform on $[-1/2, 1/2]$, so $\mathbb{P}(|\delta_i| \le \delta_0) = 2\delta_0$. Independence gives the product. □

Equivalently, an inverse check infers μ_0 from a near-integer depth assignment. Let $r_*(\mu) := \text{round}(r(\mu))$ and define $\mu_0(\mu) := \mu/\varphi^{r_*(\mu)}$. For electroweak and leptonic reference scales, the inferred values cluster around m_e:

Proposition H.3 (Multiplicative minimax centering of the reference scale)**.** *Fix scales* $\{\mu_i\}_{i=1}^N$ *and an integer depth assignment* $\{r_i^*\}_{i=1}^N$*. Define the implied reference scales*

$$\mu_{0,i} := \frac{\mu_i}{\varphi^{r_i^*}}. \tag{121}$$

Then the choice

$$\mu_0^* := \sqrt{\min_i \mu_{0,i}\ \max_i \mu_{0,i}} \tag{122}$$

Scale μ	$\mu\,[\mathrm{GeV}]$	$r(\mu)$	$r - \mathrm{round}(r)$
$\Lambda^{(5)}_{\overline{\mathrm{MS}}}$	2.09×10^{-1}	12.497	+0.497
m_c	1.27	16.247	+0.247
m_b	4.18	18.722	−0.278
m_t	1.7276×10^{2}	26.456	+0.456

Table 9: Extended calibration of the Fibonacci resolution coordinate $r(\mu) = \log(\mu/m_e)/\log\varphi$ for additional QCD/heavy-flavor scales, using standard reference values [24].

uniquely minimizes the multiplicative worst-case deviation $\max_i \big|\log(\mu_{0,i}/\mu_0)\big|$. *Equivalently,* μ_0^* *uniquely minimizes* $\max_i |r(\mu_i) - r_i^*|$ *since*

$$r(\mu_i) - r_i^* = \frac{\log(\mu_{0,i}/\mu_0)}{\log\varphi}. \tag{123}$$

For $\{\mu_i\} = \{m_\mu, m_\tau, m_W, m_Z\}$ *with* $(r_\mu^*, r_\tau^*, r_W^*, r_Z^*) = (11, 17, 25, 25)$ *one finds*

$$\mu_0^* \approx 5.1032 \times 10^{-4}\,\mathrm{GeV} = 0.998672\,m_e, \tag{124}$$

and the resulting maximal depth deviation decreases from $\delta_0 = 0.133880$ *(Table 7) to* $\delta_0^* = 0.131118$*, giving* $(2\delta_0^*)^4 \approx 4.73 \times 10^{-3}$ *under the same uniform null. Thus the rigid convention* $\mu_0 = m_e$ *lies within* 0.13% *of the multiplicative minimax center for this electroweak/leptonic anchor set.*

Proof. For $\mu_0 > 0$ define $t := \log\mu_0$ and $t_i := \log\mu_{0,i}$. Then $\max_i |\log(\mu_{0,i}/\mu_0)| = \max_i |t_i - t|$. This is minimized uniquely at the midrange $t^* = (\min_i t_i + \max_i t_i)/2$, i.e. $\mu_0^* = \exp(t^*) = \sqrt{\min_i \mu_{0,i}\,\max_i \mu_{0,i}}$. The equivalence with minimizing $\max_i |r(\mu_i) - r_i^*|$ follows by the displayed identity. □

Table 9 lists additional QCD/heavy-flavor reference scales. These are scheme-dependent and enter as matching inputs rather than as integer-depth anchors.

Proposition H.4 (Scheme rescaling induces an additive depth shift)**.** *Define*

$$r(\mu) := \frac{\log(\mu/m_e)}{\log\varphi}.$$

If a convention or renormalization scheme rescales a reference scale multiplicatively, $\mu' = s\,\mu$ *with* $s > 0$*, then*

$$r(\mu') = r(\mu) + \frac{\log s}{\log\varphi}. \tag{125}$$

In particular, the fractional part of $r(\mu)$ *is not invariant under such rescalings. Near-integer depth tests are meaningful only for scheme-stable anchor scales; scheme-dependent scales are treated as matching inputs.*

Proof. This is immediate from the definition:

$$r(\mu') = \frac{\log(s\mu/m_e)}{\log\varphi} = \frac{\log(\mu/m_e)}{\log\varphi} + \frac{\log s}{\log\varphi}.$$

□

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