The Motive at Infinity: Functorialization of the Holographic Scanning Principle, Period Realizations, and a Selection Principle

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Abstract

We continue an audit-driven program in which finite-resolution scanning and readout are treated as primary mathematical structure, rather than postulates external to an "ontic" dynamics. Building on an earlier closed pipeline from noncommutative scanning to cusp coefficient spectra, Hecke dynamics, and Langlands semantics, we introduce an additional layer "above the cusp": a period–motive interface that explains why stable constants repeatedly appear as π, log, and zeta-values.

On a controlled class of protocols we define a category $\mathsf{Scan}_{\mathrm{alg}}$ whose objects are Kronecker-type scans on tori equipped with algebraic (rational) readout kernels and auditable regularization rules. We construct a *holographic scanning functor* $\mathrm{HSP} : \mathsf{Scan}_{\mathrm{alg}} \to \mathsf{PerDatum}$ into a category of period data and prove a closed realization theorem: for rationally independent scan slopes, the long-time Birkhoff readout equals the Kontsevich–Zagier period associated with $\mathrm{HSP}(\mathcal{P})$. Finite-resource implementations admit an explicit auditable error decomposition into (i) a sampling discrepancy term controlled by diophantine properties of the scan and (ii) a truncation/regularization term with provable bounds. In particular, for truncated geometric kernels realizing $\zeta(d)$, the error budget yields a closed stability–truncation tradeoff and an explicit optimized truncation-depth choice under any discrepancy certificate.

We provide reproducible pure-Python experiments realizing $\log 2$, π, $\zeta(2)$ and $\zeta(3)$ via one-, two-, and three-dimensional scans using truncated geometric kernels. As a toy model for "constant selection" under a bounded description budget, we reproduce a low-complexity search in which $4\pi^3 + \pi^2 + \pi$ emerges as the unique best approximation to α^{-1} among nonnegative integer combinations of π, π^2, π^3 with bounded coefficient sum. Finally, we formulate a falsifiable selection principle: observable constants correspond to period data that minimize a stability–complexity functional under finite resource constraints, motivating a minimal operative notion of *protocol-stable period data* as protocol-invariant, stably realizable period structures.

Keywords: holographic scanning principle; Kronecker scan; equidistribution; discrepancy; Koksma–Hlawka inequality; Kontsevich–Zagier periods; motives; multiple zeta values; regularization; error budget; functoriality; selection principle.

Conventions. Unless otherwise stated, log denotes the natural logarithm. "mod 1" refers to reduction in $\mathbb{R}/\mathbb{Z}$. The scan time $t \in \mathbb{Z}_{\geq 0}$ denotes an iteration count of a protocol-defined update (Layer 1). We identify the d-torus with $\mathbb{T}^d = (\mathbb{R}/\mathbb{Z})^d$ and use the fundamental domain $[0,1)^d$ when writing integrals. When convenient, we freely replace $[0,1)^d$ by $[0,1]^d$ in integrals and discrepancy definitions, since the boundary has Lebesgue measure zero.

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Contents

1 Introduction: from the stairway to the motive

Foundational mathematical physics faces a persistent mismatch between what is *modeled* and what is *recorded.* Microscopic descriptions privilege unitary evolution and continuous symmetries; observational records are discrete, finite-resolution, and statistical. The standard narrative resolves this by placing continuity in "ontology" and discreteness in an externally imposed measurement postulate. In the audit program pursued across the HPA–Ω manuscripts, the organizational stance is the opposite: *readout is part of the structure.* Time, probability, and discreteness must arise within an explicit scan–readout protocol under finite information constraints, in a way that admits a concrete dependency chain (Layer 0/1) and a strict separation from semantic narratives (Layer 2).

In a companion work, [1] ("The Stairway to Infinity") formalized the climb from noncommutative scanning to arithmetic rigidity and further to automorphic (Langlands) semantics. The paper closed an auditable chain

$$\begin{aligned}\text{Weyl pair} \implies A_\alpha \implies &\text{ (modular) geodesic/Gauss flow}\\ \implies &\text{ cusp } q\text{-coefficients} \implies \text{Hecke prime skeleton,}\end{aligned} \tag{1}$$

and left two explicit tasks open: a functorial upgrade of protocol-level constructions and a principled equivalence criterion for protocols.

The present paper addresses a different, more "source-level" question that remains after (1) is closed:

- Why do stable constants persistently reappear as π, logarithms, and zeta-values?
- Why does the modular/Hecke route look canonical rather than contingent?
- What is the structural origin of the arithmetic rigidity observed at cusps, beyond the level of coefficient extraction procedures?

Our answer is not a new physical narrative; it is a further audit-compatible *mathematical* move. We introduce a period–motive interface as a minimal structure "above the cusp" and show that, in a controlled subcategory, scanning itself is an algorithmic realization of Kontsevich–Zagier periods.

1.1 Core claim: scanning as a period realization

The key thesis is closed in the controlled protocol class $\mathsf{Scan}_{\mathrm{alg}}$ (Section 4) and can be stated as a single theorem by combining the realization results of Section 5 with the finite-resource bounds of Section 6.

Theorem 1.1 (Scanning realizes periods with an auditable finite-N certificate)**.** *Let* $\mathcal{P} = (\mathbb{T}^d, \alpha, x_0, f, \mathcal{R}) \in \mathsf{Scan}_{\mathrm{alg}}$ *and let* $P_N = \{x_0, \ldots, x_{N-1}\} \subset [0,1)^d$ *be its orbit-prefix point set. Assume the rational-independence condition* (21) *(or* $\alpha \notin \mathbb{Q}$ *if* $d = 1$*) and that the regularized kernel* $f_{\mathcal{R}}$ *is Riemann integrable on* $[0,1)^d$*. Then*

$$\lim_{N\to\infty} \langle \mathcal{P} \rangle_N = \mathrm{per}(\mathrm{HSP}(\mathcal{P})) = \int_{[0,1)^d} f_{\mathcal{R}}(x)\,\mathrm{d}x. \tag{2}$$

If, moreover, $\mathrm{Var}_{\mathrm{HK}}(f_{\mathcal{R}}) < \infty$*, then for every* $N \geq 1$ *one has the explicit finite-resource bound*

$$|\langle \mathcal{P} \rangle_N - \mathrm{per}(\mathrm{HSP}(\mathcal{P}))| \leq \mathrm{Var}_{\mathrm{HK}}(f_{\mathcal{R}})\, D_N^*(P_N). \tag{3}$$

Finally, if f *is an ideal kernel and* $\mathcal{R}$ *is a regularization rule, then the total error to the ideal period admits an auditable sampling–regularization decomposition (Proposition 6.7).*

Proof. The limit statement (2) is Corollary 5.4 (or Corollary 5.2 in $d = 1$). The finite-N certificate (3) is Corollary 6.3. The final sentence is Proposition 6.7. ☐

Theorem 1.1 has two consequences.

- **Numerical layer.** The repeated appearance of π, log, and zeta-values is explained by the fact that these constants are periods, and the protocol computes periods.
- **Structural layer.** "Protocol equivalence" can be anchored to period data: morphisms in $\mathsf{Scan}_{\mathrm{alg}}$ induce period-datum equivalences in $\mathsf{PerDatum}$, hence $\mathsf{Scan}_{\mathrm{alg}}$-isomorphic protocols yield identical long-time limits and identical period values (Section 4). A full classification of protocols by period data is left as a separate problem.

1.2 What is new in this paper

Compared to [1], this paper contributes the following closed components.

- **A controlled protocol category $\mathsf{Scan}_{\mathrm{alg}}$.** We define a subcategory of protocols whose scan dynamics are Kronecker rotations on tori and whose readout kernels are rational functions on $[0, 1)^d$, equipped with explicit regularization rules (Section 4).
- **A period-data category $\mathsf{PerDatum}$ and a functor HSP.** We define $\mathsf{PerDatum}$ as a category of period data and construct a holographic scanning functor $\mathrm{HSP} : \mathsf{Scan}_{\mathrm{alg}} \to \mathsf{PerDatum}$ mapping protocols to integrable rational kernels on $[0, 1)^d$ (Section 4).
- **A closed realization theorem.** We prove that, under the standard rational-independence condition on the scan slope, the Birkhoff readout average converges to the period associated with $\mathrm{HSP}(\mathcal{P})$ (Section 5).
- **Auditable error budgets.** We provide explicit finite-N error decompositions into discrepancy-controlled sampling error and provable truncation/regularization error, and we emphasize how these bounds propagate across protocol compositions (Section 6).
- **Reproducible experiments and a toy selection signal.** We provide pure-Python scripts reproducing period realizations of $\log 2$, π, $\zeta(2)$ and $\zeta(3)$ and a low-complexity constant search illustrating a sharp "uniqueness gap" under a description budget (Section 9).

Beyond the closed components, we formulate a falsifiable *selection principle* as a programmatic statement: under finite resources, period data that are stably realizable with low description complexity are preferentially selected. This principle is presented as a conjectural interface (Section 7) and does not enter the closed proof chain.

1.3 Relation to quasi–Monte Carlo and Kronecker sequences

The deterministic finite-N certificate (3) is classical in quasi–Monte Carlo (QMC) integration: Koksma–Hlawka bounds the integration error of a point set by the product of a variation seminorm and a discrepancy seminorm [2–4]. Within that literature, Kronecker point sets and their shifted variants are standard examples, closely related to rank-1 lattice rules: a generating vector (here the scan slope α) produces a structured orbit on the torus, and translation by an initial point x_0 corresponds to a shift [4, 5]. Discrepancy control for such constructions is classically expressed via Erdős–Turán–Koksma inequalities and Diophantine approximation properties of α [2–4, 6].

The emphasis of this paper is not to optimize discrepancy constants against the best available QMC constructions, nor to replace randomized QMC methodology. Rather, we treat the point set as an explicit *protocol output* generated by a simple dynamical rule, and we require that both the kernel regularization and the discrepancy certificate be auditable within the protocol. This motivates: (i) working with Kronecker scans (simple, parameterized by α), (ii) recording explicit ETK-type certificates with explicit constants (Appendix B.5), and (iii) using regularizations (such as the truncated geometric kernels for $\zeta(d)$) that yield finite Hardy–Krause variation with explicit bounds (Section 6).

1.4 Relation to motives and to the Langlands chain

Motives organize the relations among periods by comparing different cohomological realizations (Betti, de Rham, ℓ-adic). We do not assume any conjectural statements from the general theory of motives. The role of motives in this paper is *organizational* and appears only through a minimal "period datum" interface that stays within Layer 0/1 audit constraints.

Nevertheless, the motive viewpoint offers a natural compatibility target for the stairway chain: Hecke eigenvalues can be interpreted as Frobenius traces on suitable realizations of motives, while periods are the numerical shadows of comparison isomorphisms. Section 8 formulates a commutative-diagram objective that factors the functorial Langlands upgrade of [1] through period data and motives.

Outline. Section 2 fixes the audit-layer conventions and clarifies how we use motives minimally. Section 3 recalls periods and introduces a minimal notion of protocol-stable period data. Section 4 defines $\mathsf{Scan}_{\mathrm{alg}}$, $\mathsf{PerDatum}$, and the functor HSP. Section 5 proves the main realization theorem. Section 6 develops error budgets. Section 7 proposes the selection principle. Section 8 discusses compatibility with the Langlands chain. Section 9 provides reproducible experiments. We conclude in Section 10.

2 Audit rules and layer discipline

The audit program enforces a strict separation between what is assumed, what is derived, and what is merely interpreted. This paper follows the same discipline as [1, 7, 8].

2.1 Layer conventions

Layer 0 (ontic). Only algebraic objects and states are allowed. No external time parameter, probability postulate, or observer semantics is assumed. In the present paper, Layer 0 objects are the underlying dynamical system (a torus rotation), the readout kernel, and the invariant measure.

Layer 1 (protocol). A protocol specifies how "time" is realized as an iteration count, how finite-resolution readout is implemented, and how statistics are induced from repeated readout. In this paper, "time" is the scan index $t \in \mathbb{Z}_{\geq 0}$, and probabilities arise from empirical distributions along scan orbits (Birkhoff averages) and, optionally, from finite ensembles of initial conditions used only as a numerical stabilization device.

Layer 2 (interpretation). Any physical semantics (spacetime, particles, gravity narratives) is permitted only as commentary. It may not appear as a premise in Layer 0/1 arguments. This paper stays in Layer 0/1 for the closed proof chain; interpretation-level language is avoided in theorems and proofs.

2.2 A minimal "Layer -1" interface

Motives and their realization theories form a deep and wide subject. To keep the present paper auditable, we adopt a strict strategy.

- All closed statements in the main text use only the *period* layer: integrals of rational functions on semialgebraic domains (Kontsevich–Zagier periods). This layer admits explicit definitions and stability properties required for finite-resolution protocols.
- Motives appear only as *organizational language* to name a potential source object behind multiple realizations of the same period data. No unproven conjecture about motives is used as an input.
- Whenever a statement uses motive-level terminology in a programmatic way (e.g. a comparison with ℓ-adic realizations), it is explicitly labeled as conjectural/programmatic and never used in the closed dependency chain.

In this sense, the paper uses a minimal "Layer -1" interface: not as a foundation beneath Layer 0/1, but as a controlled vocabulary for organizing where period data might come from and how it could connect to the Langlands chain.

2.3 Audit status classification

To make the boundary of the closed content explicit, Appendix A provides a status classification:

- **Standard facts**: results from equidistribution and discrepancy theory (Weyl/Kronecker equidistribution, Koksma–Hlawka inequality).
- **New definitions**: $\mathsf{Scan}_{\mathrm{alg}}$, $\mathsf{PerDatum}$, HSP, and protocol-stable period data (Sections 3–4).
- **Closed conclusions**: the realization theorem (Section 5) and the auditable error budget decomposition (Section 6).
- **Programmatic conjectures**: the selection principle (Section 7) and the motive/Langlands commutative-diagram objective (Section 8).

3 Periods, computability, and protocol-stable period data

This section recalls the period layer that will serve as our auditable "interface" to motives and introduces a minimal, protocol-based stability notion for period data. Throughout, we keep the closed proof chain within the period layer and use motive language only to indicate a possible source structure for period relations.

3.1 Kontsevich–Zagier periods

Definition 3.1 (Kontsevich–Zagier period)**.** *A complex number $P \in \mathbb{C}$ is a* (Kontsevich–Zagier) period *if there exist*

- *an integer $n \geq 1$,*
- *a semialgebraic domain $D \subset \mathbb{R}^n$ defined by finitely many polynomial inequalities with coefficients in $\mathbb{Q}$, and*
- *a rational function $f(x) \in \mathbb{Q}(x_1, \ldots, x_n)$*

such that the integral converges absolutely and

$$P = \int_D f(x)\, \mathrm{d}x_1 \cdots \mathrm{d}x_n. \tag{4}$$

Kontsevich–Zagier periods form a countable $\mathbb{Q}$-algebra under addition and multiplication [9]. They contain classical constants such as π and $\log 2$, and encompass many special values arising in arithmetic geometry and perturbative quantum field theory [9, 10].

Remark 3.2 (Controlled scope used in this paper)**.** *For auditable protocol realizations, we will primarily use* cubical *representations with $D = [0,1]^d$ and f an integrable rational function on D. This already captures the examples relevant for the experiments in Section 9 and is stable under the discrepancy-based error analysis of Section 6.*

3.2 Period data as auditable objects

The KZ definition characterizes numbers. For functorialization, we need an object-level description.

Definition 3.3 (Period datum)**.** *A* period datum *is a triple $(D, f, \mathcal{R})$ where:*

- *$D \subset \mathbb{R}^d$ is a $\mathbb{Q}$-semialgebraic domain,*
- *$f \in \mathbb{Q}(x_1, \ldots, x_d)$ is a rational function that is integrable on D after applying a regularization rule $\mathcal{R}$ (possibly the identity),*
- *$\mathcal{R}$ specifies how to interpret f if it is presented via a convergent/regularized expression (e.g. truncation of a series, principal value conventions, or a limit along a prescribed subsequence), producing an integrable kernel $f_{\mathcal{R}}$ on D.*

The associated numerical period is

$$\mathrm{per}(D, f, \mathcal{R}) := \int_D f_{\mathcal{R}}(x)\,\mathrm{d}x. \tag{5}$$

Remark 3.4 (When a period datum yields a Kontsevich–Zagier period)**.** *Definition 3.3 is an object-level encoding of an auditable integral value. If, moreover, D is $\mathbb{Q}$-semialgebraic and the regularized kernel $f_{\mathcal{R}}$ is (piecewise) rational with coefficients in $\mathbb{Q}$ on a finite $\mathbb{Q}$-semialgebraic partition of D* and the integral converges absolutely *(equivalently, $f_{\mathcal{R}} \in L^1(D)$), then* $\mathrm{per}(D, f, \mathcal{R})$ *is a Kontsevich–Zagier period in the sense of Definition 3.1 (by additivity over the partition). All kernels used in the closed chain of this paper fall in this class: the one-dimensional benchmarks are rational, and the truncations used for zeta-values produce polynomial (hence rational) kernels.*

Remark 3.5 (Motivic background, not used as an axiom)**.** *In the standard paradigm, periods arise as pairings between de Rham cohomology classes and Betti homology classes (or, more generally, between two realizations of the same motive). Relations among periods are governed by the structure of motives and comparison isomorphisms [11, 12]. This paper does not assume any conjectural principle about periods or motives; we use the period datum only as an auditable intermediate object.*

3.3 A minimal operative notion of protocol-stable period data

The word "motive" suggests an origin. To keep the stability notion operational and falsifiable, we define protocol-stable period data purely through finite-resolution realizability and protocol invariance.

Remark 3.6 (Motive vs. period: what is constructed here)**.** *In standard arithmetic geometry, a* motive *is an abstract source object whose realizations (Betti, de Rham, ℓ-adic) produce, among other invariants, periods as numerical comparison data. In this paper we do* not *construct a motive-valued functor from protocols, and we do not assume any period conjecture. The only closed construction is the functor* HSP : $\mathsf{Scan}_{\mathrm{alg}} \to \mathsf{PerDatum}$ *(Section 4), i.e. a* period realization *interface.*

Definition 3.7 (Protocol-stable period datum). *Fix a class of protocols* $\mathsf{Scan}_{\text{alg}}$ *(defined in Section 4). A period datum* $\mathcal{D} = (D, f, \mathcal{R})$ *is called a* protocol-stable period datum *(relative to* $\mathsf{Scan}_{\text{alg}}$*) if there exists a family of protocols* $\{\mathcal{P}_\lambda\}_{\lambda\in\Lambda} \subset \mathsf{Scan}_{\text{alg}}$ *and an auditable error budget* $\varepsilon(N, \lambda) \to 0$ *as* $N \to \infty$ *such that:*

- *for each* λ*, the protocol produces an empirical readout* $\langle \mathcal{P}_\lambda \rangle_N$ *(Definition 4.7) satisfying*

$$\big|\langle \mathcal{P}_\lambda \rangle_N - \mathrm{per}(\mathcal{D})\big| \le \varepsilon(N, \lambda) \quad \textit{for all } N, \tag{6}$$

- *the value* $\mathrm{per}(\mathcal{D})$ *is invariant under the protocol equivalence relation in* $\mathsf{Scan}_{\text{alg}}$ *(Section 4.5), i.e. equivalent protocols realize the same period datum up to the morphisms of* $\mathsf{PerDatum}$*.*

Definition 3.7 does not identify a protocol-stable period datum with "a number" alone; it identifies a stable period *structure* that survives finite-resolution realization and protocol equivalence. The selection principle in Section 7 will treat protocol-stable period data as candidates that are preferentially realized under bounded resources.

4 From protocols to functors: $\mathsf{Scan}_{\text{alg}}$, $\mathsf{PerDatum}$, and the holographic scanning functor

We now formalize a controlled category of scan–readout protocols and construct a functor to period data. The guiding principle is to stay within a class where both the long-time limit (equidistribution) and finite-resource error control (discrepancy) are available.

4.1 The controlled protocol category $\mathsf{Scan}_{\text{alg}}$

Definition 4.1 ($\mathsf{Scan}_{\text{alg}}$ objects). *An object of* $\mathsf{Scan}_{\text{alg}}$ *is a quintuple*

$$\mathcal{P} = (\mathbb{T}^d, \alpha, x_0, f, \mathcal{R}), \tag{7}$$

where:

- $\mathbb{T}^d = (\mathbb{R}/\mathbb{Z})^d$ *is the mother space,*
- $\alpha \in \mathbb{R}^d$ *is a scan slope (Kronecker rotation vector),*
- $x_0 \in \mathbb{T}^d$ *is an initial condition,*
- f *is a readout kernel presented as a rational function in* $\mathbb{Q}(x_1, \dots, x_d)$ *on the fundamental domain* $[0, 1)^d$*, understood on the complement of its pole locus (a semialgebraic set of measure zero), and*
- $\mathcal{R}$ *is a regularization rule that produces a kernel* $f_{\mathcal{R}}$ *that is integrable on* $[0, 1)^d$ *(and, in the finite-resource regime of Section 6, has finite Hardy–Krause variation when discrepancy bounds are invoked).*

The protocol dynamics is the Kronecker scan

$$x_t = x_0 + t\alpha \pmod 1, \qquad t \in \mathbb{Z}_{\ge 0}. \tag{8}$$

Remark 4.2 (Regularization as an auditable component). *In many period examples, the "ideal" integrand is unbounded (e.g.* $1/(1 - xy)$ *on* $[0, 1]^2$*) even though the integral converges. To keep finite-resource error budgets auditable, we treat regularization as part of the protocol object:* $\mathcal{R}$ *specifies a concrete, reproducible transformation of the presentation into an integrable rational kernel. Section 9 uses truncations that yield explicit bounds.*

Definition 4.3 ($\mathsf{Scan}_{\mathrm{alg}}$ morphisms)**.** *Let $\mathcal{P} = (\mathbb{T}^d, \alpha, x_0, f, \mathcal{R})$ and $\mathcal{P}' = (\mathbb{T}^d, \alpha', x_0', f', \mathcal{R}')$ be objects of the same dimension d. A morphism $\varphi : \mathcal{P} \to \mathcal{P}'$ is a torus automorphism of the form*

$$\varphi(x) = Mx + b \pmod 1, \tag{9}$$

with $M \in \mathrm{SL}_d(\mathbb{Z})$ and $b \in (\mathbb{Q}/\mathbb{Z})^d$ (a rational translation), such that:

$$\alpha' = M\alpha, \qquad x_0' = \varphi(x_0), \qquad f'_{\mathcal{R}'} = f_{\mathcal{R}} \circ \varphi^{-1} \ \ \text{a.e. on } [0,1)^d. \tag{10}$$

Proposition 4.4 (Rational translations and closure into $\mathsf{PerDatum}$)**.** *Let $\varphi(x) = Mx + b \pmod 1$ with $M \in \mathrm{SL}_d(\mathbb{Z})$ and $b \in (\mathbb{Q}/\mathbb{Z})^d$. Choose representatives of b in $[0,1)^d$ and consider the induced map on the fundamental domain*

$$\psi : [0,1)^d \to [0,1)^d, \qquad \psi(x) = \{Mx + b\}.$$

Then ψ admits a finite $\mathbb{Q}$-semialgebraic partition $[0,1)^d = \bigsqcup_i D_i$ such that for each i there exists $k_i \in \mathbb{Z}^d$ with

$$\psi(x) = Mx + b - k_i \quad \text{for all } x \in D_i.$$

In particular, each restriction $\psi|_{D_i}$ is a C^1-diffeomorphism defined over $\mathbb{Q}$ with $|\det D\psi| = 1$, so ψ defines a morphism in $\mathsf{PerDatum}$ in the sense of Definition 4.11.

Proof. Write $\psi(x) = Mx + b - n(x)$ where $n(x) \in \mathbb{Z}^d$ is the unique integer vector such that each coordinate of $\psi(x)$ lies in $[0,1)$. The vector $n(x)$ is locally constant and can change only when some coordinate of $Mx+b$ crosses an integer, i.e. along hyperplanes of the form $\langle e_j, Mx\rangle = m - b_j$ with $m \in \mathbb{Z}$. Over the bounded domain $[0,1)^d$ only finitely many such hyperplanes occur. If $b \in (\mathbb{Q}/\mathbb{Z})^d$, these hyperplanes have rational offsets, hence they induce a finite polyhedral partition by $\mathbb{Q}$-semialgebraic sets. On each cell D_i the vector $n(x)$ is constant, so ψ is affine with rational coefficients and Jacobian M. □

Example 4.5 (Why irrational translations are excluded in $\mathsf{Scan}_{\mathrm{alg}}$)**.** *In $d = 1$ with $M = 1$, the torus translation $\varphi(x) = x + b \pmod 1$ corresponds on $[0,1)$ to*

$$\psi(x) = \{x + b\} = \begin{cases} x + b, & 0 \le x < 1 - b, \\ x + b - 1, & 1 - b \le x < 1. \end{cases}$$

If $b \notin (\mathbb{Q}/\mathbb{Z})$, then the breakpoint $1 - b$ is irrational, so this piecewise-affine representation does not admit a finite partition into $\mathbb{Q}$-semialgebraic sets on which ψ is defined over $\mathbb{Q}$. Such translations therefore fall outside the controlled morphism class in Definition 4.11.

Remark 4.6 (What would change if one allowed arbitrary $b \in \mathbb{T}^d$?)**.** *Allowing arbitrary b would not affect measure preservation on $\mathbb{T}^d$ and hence would not break the invariance statements proved later. The rationality restriction is imposed to keep the functor* HSP *closed into the $\mathbb{Q}$-definable category* $\mathsf{PerDatum}$ *used in this paper, so that morphisms are encoded by finite rational data.*

Morphisms in Definition 4.3 are *auditable* in the sense that they explicitly intertwine scan dynamics and readout kernels while preserving Lebesgue measure on $\mathbb{T}^d$ (since $\det M = 1$). This is sufficient for the functorial invariance statements we need.

4.2 Protocol outputs: Birkhoff averages and ensembles

Definition 4.7 (Birkhoff readout average)**.** *For a protocol $\mathcal{P}$ and a horizon $N \ge 1$, define the N-step readout average*

$$\langle \mathcal{P} \rangle_N := \frac{1}{N} \sum_{t=0}^{N-1} f_{\mathcal{R}}(x_t), \qquad x_t = x_0 + t\alpha \pmod 1. \tag{11}$$

For numerical stability, one may also average over a finite ensemble of initial conditions. This does not change the theoretical limit under unique ergodicity but reduces variance in finite-N experiments.

Definition 4.8 (Finite ensemble mean). *Let $\{x_0^{(k)}\}_{k=1}^K \subset \mathbb{T}^d$ be K initial conditions, and let $\mathcal{P}^{(k)}$ denote the protocol with the same $(d, \alpha, f, \mathcal{R})$ but initial point $x_0^{(k)}$. Define*

$$\overline{\langle \mathcal{P} \rangle_N}_K := \frac{1}{K} \sum_{k=1}^K \langle \mathcal{P}^{(k)} \rangle_N. \tag{12}$$

Lemma 4.9 (Ensemble averaging preserves deterministic error certificates). *Suppose $|\langle \mathcal{P}^{(k)} \rangle_N - P| \le \varepsilon_k$ for $k = 1, \dots, K$, where $P \in \mathbb{R}$ is a target value and $\varepsilon_k \ge 0$ are bounds. Then*

$$\left| \overline{\langle \mathcal{P} \rangle_N}_K - P \right| \le \frac{1}{K} \sum_{k=1}^K \varepsilon_k.$$

In particular, if a uniform bound $|\langle \mathcal{P}^{(k)} \rangle_N - P| \le \varepsilon$ holds for all k, then $|\overline{\langle \mathcal{P} \rangle_N}_K - P| \le \varepsilon$.

Proof. Triangle inequality. □

4.3 The period-data category PerDatum

Definition 4.10 (PerDatum objects). *An object of* PerDatum *is a period datum $(D, f, \mathcal{R})$ as in Definition 3.3. In this paper we will mostly use the cubical case $D = [0, 1)^d$.*

Definition 4.11 (PerDatum morphisms). *A morphism $(D, f, \mathcal{R}) \to (D', f', \mathcal{R}')$ is a map $\psi : D \to D'$ for which there exists a finite semialgebraic partition $D = \bigsqcup_{i=1}^m D_i$ such that each restriction $\psi|_{D_i} : D_i \to \psi(D_i)$ is a C^1-diffeomorphism defined over $\mathbb{Q}$ and the change-of-variables identity holds almost everywhere:*

$$f_{\mathcal{R}}(x)\, \mathrm{d}x = f'_{\mathcal{R}'}(\psi(x)) \left| \det D\psi(x) \right| \mathrm{d}x, \tag{13}$$

so that $\mathrm{per}(D, f, \mathcal{R}) = \mathrm{per}(D', f', \mathcal{R}')$.

Definition 4.11 encodes the idea that period data should be considered up to auditable changes of variables and regularization conventions.

4.4 The holographic scanning functor HSP

Definition 4.12 (Holographic scanning functor). *Define a functor*

$$\mathrm{HSP} : \mathsf{Scan}_{\mathrm{alg}} \to \mathsf{PerDatum} \tag{14}$$

as follows.

- *On objects: for $\mathcal{P} = (\mathbb{T}^d, \alpha, x_0, f, \mathcal{R})$, set*

$$\mathrm{HSP}(\mathcal{P}) := ([0, 1)^d,\ f,\ \mathcal{R}). \tag{15}$$

- *On morphisms: if $\varphi : \mathcal{P} \to \mathcal{P}'$ is a* $\mathsf{Scan}_{\mathrm{alg}}$ *morphism, define* $\mathrm{HSP}(\varphi)$ *to be the induced change of variables on fundamental domains, viewed as a* PerDatum *morphism in the sense of Definition 4.11.*

Remark 4.13 (Fundamental domains and a.e. maps)**.** *The torus automorphism* $\varphi(x) = Mx+b \pmod 1$ *is smooth on* $\mathbb{T}^d$ *but, when represented on the fundamental domain* $[0,1)^d$*, it becomes a piecewise-affine map* $x \mapsto \{Mx+b\}$ *with discontinuities only along a finite union of codimension-*1 *hyperplanes coming from wrap-around. Since these discontinuity sets have Lebesgue measure zero, the "a.e." formulation in Definition 4.3 and the piecewise-*C^1 *formulation in Definition 4.11 make* $\mathrm{HSP}(\varphi)$ *well-defined and auditable.*

Lemma 4.14 (Category structure and functoriality)**.** *With Definitions 4.3 and 4.11,* $\mathsf{Scan}_{\mathrm{alg}}$ *and* $\mathsf{PerDatum}$ *are categories, and* HSP *(Definition 4.12) is a well-defined functor.*

Proof. $\mathsf{Scan}_{\mathrm{alg}}$**.** Composition of torus automorphisms (M_1, b_1) and (M_2, b_2) is $(M_2M_1, M_2b_1+b_2)$; if $b_1, b_2 \in (\mathbb{Q}/\mathbb{Z})^d$ then $M_2b_1 + b_2 \in (\mathbb{Q}/\mathbb{Z})^d$ since $M_2 \in \mathrm{SL}_d(\mathbb{Z})$. The intertwining conditions (10) are preserved under composition, and the identity morphism is $(I, 0)$.

$\mathsf{PerDatum}$**.** Identity morphisms are given by $\psi = \mathrm{id}$. If $\psi : D \to D'$ and $\psi' : D' \to D''$ are piecewise-C^1 semialgebraic morphisms over $\mathbb{Q}$, then $\psi' \circ \psi$ is piecewise-C^1 on a finite semialgebraic refinement of the domain partition, and the change-of-variables identity (13) composes on each piece (hence a.e. on D).

HSP**.** On objects, HSP is defined by (15). On morphisms, $\mathrm{HSP}(\varphi)$ is well-defined by Remark 4.13 and satisfies $\mathrm{HSP}(\mathrm{id}) = \mathrm{id}$ and $\mathrm{HSP}(\varphi_2 \circ \varphi_1) = \mathrm{HSP}(\varphi_2) \circ \mathrm{HSP}(\varphi_1)$ because both are induced by the same underlying torus automorphisms and their compositions. ☐

Proposition 4.15 (Functorial invariance of readout averages)**.** *If* $\varphi : \mathcal{P} \to \mathcal{P}'$ *is a morphism in* $\mathsf{Scan}_{\mathrm{alg}}$*, then for every* $N \geq 1$*,*

$$\langle \mathcal{P} \rangle_N = \langle \mathcal{P}' \rangle_N, \tag{16}$$

and the associated period values agree:

$$\mathrm{per}(\mathrm{HSP}(\mathcal{P})) = \mathrm{per}(\mathrm{HSP}(\mathcal{P}')). \tag{17}$$

Proof. The intertwining conditions (10) imply $\varphi(x_t) = x'_t$ for all t and $f'_{\mathcal{R}'}(x'_t) = f_{\mathcal{R}}(x_t)$, hence (16). Since φ preserves Lebesgue measure on $\mathbb{T}^d$, the change-of-variables identity gives (17). ☐

4.5 Protocol equivalence and period invariants

The previous paper [1] emphasized the need for an equivalence criterion for protocols. In the controlled setting of $\mathsf{Scan}_{\mathrm{alg}}$, we can state a minimal long-time criterion.

Definition 4.16 (Long-time statistical equivalence)**.** *Two protocols* $\mathcal{P}, \mathcal{P}' \in \mathsf{Scan}_{\mathrm{alg}}$ *are* long-time statistically equivalent*, written* $\mathcal{P} \sim_\infty \mathcal{P}'$*, if both limits exist and*

$$\lim_{N\to\infty} \langle \mathcal{P} \rangle_N = \lim_{N\to\infty} \langle \mathcal{P}' \rangle_N. \tag{18}$$

If, moreover, the common value equals $\mathrm{per}(\mathrm{HSP}(\mathcal{P})) = \mathrm{per}(\mathrm{HSP}(\mathcal{P}'))$*, we say the protocols are* period-equivalent*.*

In Sections 5 and 6 we show that, under standard rational-independence conditions on α, long-time statistics are determined by the period datum and finite-N stability is controlled by discrepancy and regularization errors.

5 Main results: Birkhoff averages realize periods

This section establishes the closed core of the paper: in the controlled protocol class $\mathsf{Scan}_{\mathrm{alg}}$, long-time readout averages agree with the period values produced by the functor HSP.

5.1 One-dimensional realization

Theorem 5.1 (Weyl equidistribution ⇒ period realization in $d = 1$)**.** *Let $\alpha \in \mathbb{R} \setminus \mathbb{Q}$ and define $x_t = x_0 + t\alpha \pmod 1$ on $\mathbb{T}^1$. If $f : [0, 1) \to \mathbb{R}$ is Riemann integrable, then*

$$\lim_{N\to\infty} \frac{1}{N} \sum_{t=0}^{N-1} f(x_t) = \int_0^1 f(x)\,\mathrm{d}x. \tag{19}$$

Proof. The sequence $(x_t)_{t\geq 0}$ is uniformly distributed modulo 1 for irrational α [2, 13]. Weyl's criterion implies (19) for all Riemann integrable f. □

Corollary 5.2 (Protocol limit equals the period value in $d = 1$)**.** *Let $\mathcal{P} = (\mathbb{T}^1, \alpha, x_0, f, \mathcal{R}) \in \mathsf{Scan}_{\mathrm{alg}}$ with $\alpha \notin \mathbb{Q}$ and suppose $f_{\mathcal{R}}$ is Riemann integrable on $[0, 1)$. Then*

$$\lim_{N\to\infty} \langle \mathcal{P} \rangle_N = \mathrm{per}(\mathrm{HSP}(\mathcal{P})) = \int_0^1 f_{\mathcal{R}}(x)\,\mathrm{d}x. \tag{20}$$

5.2 Multi-dimensional realization

Theorem 5.3 (Kronecker equidistribution ⇒ period realization)**.** *Let $\alpha \in \mathbb{R}^d$ satisfy the rational-independence condition*

$$1, \alpha_1, \ldots, \alpha_d \ \ \textit{are linearly independent over } \mathbb{Q}. \tag{21}$$

Define $x_t = x_0 + t\alpha \pmod 1$ on $\mathbb{T}^d$. If $f : [0, 1)^d \to \mathbb{R}$ is Riemann integrable, then

$$\lim_{N\to\infty} \frac{1}{N} \sum_{t=0}^{N-1} f(x_t) = \int_{[0,1)^d} f(x)\,\mathrm{d}x. \tag{22}$$

Proof. Condition (21) implies that the Kronecker sequence (x_t) is uniformly distributed on $\mathbb{T}^d$ [2]. As in the one-dimensional case, uniform distribution implies (22) for Riemann integrable f. □

Corollary 5.4 (Main realization theorem for $\mathsf{Scan}_{\mathrm{alg}}$)**.** *Let $\mathcal{P} = (\mathbb{T}^d, \alpha, x_0, f, \mathcal{R}) \in \mathsf{Scan}_{\mathrm{alg}}$ satisfy (21) and assume $f_{\mathcal{R}}$ is Riemann integrable on $[0, 1)^d$. Then*

$$\lim_{N\to\infty} \langle \mathcal{P} \rangle_N = \mathrm{per}(\mathrm{HSP}(\mathcal{P})) = \int_{[0,1)^d} f_{\mathcal{R}}(x)\,\mathrm{d}x. \tag{23}$$

Moreover, the limit is independent of x_0.

Remark 5.5 (Beyond Riemann integrability)**.** *If $f_{\mathcal{R}} \in L^1(\mathbb{T}^d)$, ergodicity of the irrational rotation under (21) implies that (23) holds for almost every initial condition x_0 by the Birkhoff ergodic theorem [14]. We restrict to Riemann integrable kernels in the closed chain because this supports explicit discrepancy-based error bounds for finite resources.*

5.3 Canonical period examples realized by scans

The following examples will be used as reproducible benchmarks in Section 9. They also illustrate why the "special constants" of the earlier papers appear naturally: they are periods with low-complexity cubical representations.

Example 5.6 ($\log 2$ as a one-dimensional scan period)**.** *Let $f(x) = \frac{1}{1+x}$ on $[0, 1)$. Then*

$$\int_0^1 \frac{1}{1+x}\,\mathrm{d}x = \log 2, \tag{24}$$

and Theorem 5.1 implies that for any irrational α,

$$\lim_{N\to\infty} \frac{1}{N} \sum_{t=0}^{N-1} \frac{1}{1 + \{x_0 + t\alpha\}} = \log 2. \tag{25}$$

Example 5.7 (π as a one-dimensional scan period)**.** *Let $f(x) = \frac{4}{1+x^2}$ on $[0,1)$. Then*

$$\int_0^1 \frac{4}{1+x^2}\,\mathrm{d}x = \pi, \tag{26}$$

and Theorem 5.1 yields a scan realization of π.

Example 5.8 (π^2 as a two-dimensional scan period)**.** *Let $f(x,y) = \dfrac{16}{(1+x^2)(1+y^2)}$ on $[0,1)^2$. Then*

$$\int_{[0,1)^2} \frac{16}{(1+x^2)(1+y^2)}\,\mathrm{d}x\,\mathrm{d}y = \left(\int_0^1 \frac{4}{1+x^2}\,\mathrm{d}x\right)^2 = \pi^2. \tag{27}$$

Under the rational-independence condition (21) in $d = 2$, Theorem 5.3 yields a scan realization of π^2 via a two-dimensional protocol.

Example 5.9 (π^3 as a three-dimensional scan period)**.** *Let $f(x,y,z) = \dfrac{64}{(1+x^2)(1+y^2)(1+z^2)}$ on $[0,1)^3$. Then*

$$\int_{[0,1)^3} \frac{64}{(1+x^2)(1+y^2)(1+z^2)}\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z = \left(\int_0^1 \frac{4}{1+x^2}\,\mathrm{d}x\right)^3 = \pi^3. \tag{28}$$

Under (21) in $d = 3$, Theorem 5.3 yields a scan realization of π^3 via a three-dimensional protocol.

Multiple zeta values are also periods [9, 10], but their most naive cubical representations involve integrands with boundary singularities. Section 6 introduces truncation regularizations that keep kernels bounded and yield explicit error bounds.

5.4 Resonant (rationally dependent) scan slopes

The rational-independence hypothesis (21) is the clean regime in which the scan explores the full torus and the limiting statistic equals the full-cube integral. When rational relations exist, the scan lives on a lower-dimensional subtorus and the limit changes accordingly. This behavior is standard and can be stated as a closed quantitative alternative to Theorem 5.3.

Theorem 5.10 (Subtorus reduction in the resonant case)**.** *Let $T_\alpha(x) = x + \alpha \pmod 1$ on $\mathbb{T}^d$ with arbitrary $\alpha \in \mathbb{R}^d$. Let $H := \overline{\{t\alpha \pmod 1 : t \in \mathbb{Z}\}}$ be the orbit-closure subgroup, which is a subtorus of $\mathbb{T}^d$, and let m_H denote its Haar probability measure. Then for every continuous function $f : \mathbb{T}^d \to \mathbb{R}$ and every $x_0 \in \mathbb{T}^d$,*

$$\lim_{N\to\infty} \frac{1}{N} \sum_{t=0}^{N-1} f(T_\alpha^t(x_0)) \;=\; \int_{x_0+H} f(x)\,\mathrm{d}m_H(x). \tag{29}$$

If $H = \mathbb{T}^d$ (equivalently, (21) holds), then (29) reduces to (22).

Lemma 5.11 (Explicit description of the orbit-closure subtorus)**.** *Define the annihilator lattice*

$$\Lambda(\alpha) := \{h \in \mathbb{Z}^d : \langle h, \alpha\rangle \in \mathbb{Z}\}.$$

Then the orbit-closure subgroup $H = \overline{\{t\alpha \pmod 1 : t \in \mathbb{Z}\}}$ satisfies

$$H = \{x \in \mathbb{T}^d : \langle h, x\rangle = 0 \pmod 1 \text{ for all } h \in \Lambda(\alpha)\}.$$

Moreover, $\Lambda(\alpha) = \{0\}$ if and only if (21) holds, in which case $H = \mathbb{T}^d$.

Proof of Lemma 5.11. Characters of $\mathbb{T}^d$ are indexed by $h \in \mathbb{Z}^d$ via $\chi_h(x) := \mathrm{e}^{2\pi\mathrm{i}\langle h,x\rangle}$. One has $\chi_h(t\alpha) = \mathrm{e}^{2\pi\mathrm{i}t\langle h,\alpha\rangle}$, so χ_h is identically 1 on the cyclic subgroup generated by α if and only if $\langle h, \alpha\rangle \in \mathbb{Z}$, i.e. $h \in \Lambda(\alpha)$. The subgroup H is the smallest closed subgroup containing α, hence it equals the intersection of kernels of all characters that vanish on α, which gives the displayed description. The final statement follows because (21) is equivalent to $\Lambda(\alpha) = \{0\}$ for translations on $\mathbb{T}^d$; see standard treatments of rotations on compact abelian groups [2,14,15]. □

Proof of Theorem 5.10. The translation T_α is uniquely ergodic on each coset of its orbit-closure subgroup; see standard treatments of rotations on compact abelian groups and Kronecker's theorem [2, 14, 15]. Unique ergodicity implies uniform convergence of Birkhoff averages for continuous observables to the Haar integral on the orbit closure, yielding (29). □

Remark 5.12 (Interpretation-layer note: "locking")**.** *When rational relations constrain H to be a proper subtorus, the protocol's long-time statistic becomes a lower-dimensional average. In interpretation-layer language this resembles phase locking or plateau behavior under resonance constraints. Theorem 5.10 is the Layer 0/1 mathematical statement behind such narratives.*

6 Finite-resource error budgets: discrepancy plus regularization

The realization theorem (Corollary 5.4) identifies the long-time limit. For finite resources, the relevant object is an auditable error budget: a bound on

$$\Big|\langle\mathcal{P}\rangle_N - \mathrm{per}(\mathrm{HSP}(\mathcal{P}))\Big| \tag{30}$$

that is explicit and propagates under protocol composition.

6.1 Sampling discrepancy and Koksma–Hlawka type bounds

Let

$$P_N := \{x_0, \dots, x_{N-1}\} \subset [0,1)^d \tag{31}$$

be a finite point set. Its star discrepancy is

$$D_N^*(P_N) := \sup_{u\in[0,1]^d} \left| \frac{1}{N}\sum_{t=0}^{N-1} \mathbf{1}_{[0,u)}(x_t) \;-\; \prod_{j=1}^{d} u_j \right|. \tag{32}$$

For brevity we write $D_N^* := D_N^*(P_N)$ when the point set is understood.

Theorem 6.1 (Koksma–Hlawka inequality)**.** *Let $f : [0,1)^d \to \mathbb{R}$ have bounded variation in the sense of Hardy–Krause, denoted $\mathrm{Var}_{\mathrm{HK}}(f) < \infty$. Then for any point set $x_0, \dots, x_{N-1} \in [0,1)^d$,*

$$\left| \frac{1}{N}\sum_{t=0}^{N-1} f(x_t) - \int_{[0,1)^d} f(x)\,\mathrm{d}x \right| \le \mathrm{Var}_{\mathrm{HK}}(f)\, D_N^*. \tag{33}$$

Remark 6.2. *We use Theorem 6.1 as a standard discrepancy-to-error bridge [2]. In $d = 1$, bounded Hardy–Krause variation reduces to classical bounded variation, and* (33) *becomes the Koksma inequality.*

For protocols $\mathcal{P} \in \mathsf{Scan}_{\mathrm{alg}}$, the point set is the scan orbit prefix $x_t = x_0 + t\alpha \pmod 1$, and f is the regularized kernel $f_{\mathcal{R}}$.

Corollary 6.3 (Sampling error bound for regular kernels)**.** *Let $\mathcal{P} = (\mathbb{T}^d, \alpha, x_0, f, \mathcal{R}) \in \mathsf{Scan}_{\mathrm{alg}}$ and assume $\mathrm{Var}_{\mathrm{HK}}(f_{\mathcal{R}}) < \infty$. Then*

$$\Big|\langle\mathcal{P}\rangle_N - \mathrm{per}(\mathrm{HSP}(\mathcal{P}))\Big| \le \mathrm{Var}_{\mathrm{HK}}(f_{\mathcal{R}})\, D_N^*(P_N). \tag{34}$$

Remark 6.4 (Discrepancy certificates for Kronecker scans)**.** *In* $\mathsf{Scan}_{\mathrm{alg}}$*, the point set* P_N *is the Kronecker orbit prefix* $x_t = x_0 + t\alpha \pmod 1$*. Appendix B.5 records the Erdős–Turán–Koksma inequality and a closed specialization to Kronecker scans, yielding the explicit finite-*N *certificate*

$$D_N^*(P_N) \le C_d \, B_{N,H}(\alpha; N),$$

valid for every integer $H \ge 1$*, with an admissible constant choice* $C_d = (3/2)^d$ *(Theorem B.5) and an explicit computable bracket term* $B_{N,H}(\alpha; N)$ *(Definition B.11).*

6.2 Regularization and truncation as auditable error terms

Many period representations involve kernels that are integrable but unbounded. To keep finite-N sampling error auditable, we adopt a two-stage realization:

- first, replace the ideal kernel f by a bounded (or finite-variation) regularized kernel $f_{\mathcal{R}}$,
- then, sample $f_{\mathcal{R}}$ along the scan orbit.

This yields a generic decomposition.

6.3 Singularity sets of measure zero and orbit avoidance

One subtlety in period realizations is that an "ideal" integrand may have a singular locus (e.g. a rational pole set) of Lebesgue measure zero. While such singularities do not affect the integral when the function is integrable, they can be a numerical hazard if a protocol samples exactly on the singular set. For Kronecker scans this risk is negligible in a precise sense.

Lemma 6.5 (Almost-sure avoidance of null singular sets)**.** *Let* $T_\alpha : \mathbb{T}^d \to \mathbb{T}^d$ *be a translation* $T_\alpha(x) = x + \alpha \pmod 1$ *and let* $S \subset \mathbb{T}^d$ *be a measurable set of Haar (Lebesgue) measure* $m(S) = 0$*. Then the set of initial conditions whose orbit hits* S *at some time,*

$$E := \{x_0 \in \mathbb{T}^d : \exists\, t \in \mathbb{Z}_{\ge 0} \text{ such that } T_\alpha^t(x_0) \in S\},$$

also satisfies $m(E) = 0$*.*

Proof. For each $t \ge 0$, the set $T_\alpha^{-t}(S)$ is a translate of S and hence has measure zero. Since

$$E = \bigcup_{t=0}^{\infty} T_\alpha^{-t}(S),$$

countable subadditivity yields $m(E) \le \sum_{t\ge 0} m(T_\alpha^{-t}(S)) = 0$. □

Remark 6.6 (Why we still regularize)**.** *Lemma 6.5 addresses the event of* hitting *the singular set. It does not control near-singular excursions, which can dominate finite-*N *error when the kernel is unbounded. For closed, auditable finite-resource bounds we therefore encode regularization/truncation as part of the protocol object (Definition 4.1) and use bounded-variation kernels whenever quantitative control is required.*

Proposition 6.7 (Auditable error decomposition)**.** *Let* f *be an "ideal" kernel on* $[0,1)^d$ *with a regularized version* $f_{\mathcal{R}}$ *integrable on* $[0,1)^d$*. For a protocol* $\mathcal{P}$ *using* $f_{\mathcal{R}}$ *at horizon* N*,*

$$\Big|\langle \mathcal{P}\rangle_N - \mathrm{per}([0,1)^d, f, \mathrm{Id})\Big| \le \underbrace{\Big|\langle \mathcal{P}\rangle_N - \mathrm{per}([0,1)^d, f, \mathcal{R})\Big|}_{\text{sampling error}} + \underbrace{\Big|\mathrm{per}([0,1)^d, f, \mathcal{R}) - \mathrm{per}([0,1)^d, f, \mathrm{Id})\Big|}_{\text{regularization error}}. \tag{35}$$

Proof. Triangle inequality. □

The second term in (35) is designed to admit a direct analytical bound. The first term is bounded by discrepancy inequalities such as (34).

Remark 6.8 (What "Id" means in $\mathrm{per}(D, f, \mathrm{Id})$)**.** *When we write* $\mathrm{per}(D, f, \mathrm{Id})$*, we mean the (Lebesgue) integral of the ideal kernel* f *on* D *whenever it exists as an absolutely integrable function, and the standard improper integral whenever* f *has integrable singularities (e.g. boundary singularities of rational kernels). Regularization rules* $\mathcal{R}$ *are introduced not to change the target value but to produce kernels with finite-variation/discrepancy certificates at finite resources.*

6.4 Truncated geometric kernels and zeta-values

We now record a regularization that will be used in the experiments: truncated geometric kernels whose integrals are generalized harmonic numbers and whose truncation error admits an explicit bound.

Definition 6.9 (Truncated geometric kernel)**.** *For* $M \in \mathbb{N}$ *define, on* $[0, 1)^d$*,*

$$g_M(x_1, \dots, x_d) := \sum_{n=0}^{M-1} \left(\prod_{j=1}^{d} x_j \right)^n . \tag{36}$$

Lemma 6.10 (Integral of truncated geometric kernel)**.** *For* $d \geq 1$ *and* $M \in \mathbb{N}$*,*

$$\int_{[0,1)^d} g_M(x)\, \mathrm{d}x = \sum_{n=1}^{M} \frac{1}{n^d} =: H_M^{(d)}. \tag{37}$$

Proof. Expand (36), use Tonelli's theorem for the finite sum, and compute

$$\int_{[0,1)^d} \left(\prod_{j=1}^{d} x_j \right)^n \mathrm{d}x = \prod_{j=1}^{d} \int_0^1 x_j^n \mathrm{d}x_j = \left(\frac{1}{n+1} \right)^d .$$

Reindex $n + 1 \mapsto n$ to obtain (37). □

Proposition 6.11 (Exact Hardy–Krause variation of g_M)**.** *For every* $d \geq 1$ *and* $M \in \mathbb{N}$*,*

$$\mathrm{Var}_{\mathrm{HK}}(g_M) = (2^d - 1)(M - 1). \tag{38}$$

Proof. For $n \geq 1$, the monomial term $\prod_{j=1}^{d} x_j^n$ has nonnegative mixed partial derivatives on $[0, 1]^d$. For functions with continuous mixed partial derivatives, the Hardy–Krause variation admits the standard representation as a sum of integrals of absolute mixed derivatives over all faces anchored at 1 (see, e.g., [3, 4]). Since all relevant derivatives are nonnegative, absolute values can be dropped. For a subset $u \subset \{1, \dots, d\}$ with $|u| = k \geq 1$, the k-fold mixed derivative of $\prod_{j=1}^{d} x_j^n$ restricted to the face where variables outside u are set to 1 equals $n^k \prod_{j \in u} x_j^{n-1}$. Integrating over $[0, 1]^k$ yields $n^k \cdot (1/n)^k = 1$. Summing over $n = 1, \dots, M-1$ gives a contribution $M-1$ for each nonempty subset u, and there are $2^d - 1$ such subsets. Summing the contributions yields (38). □

Corollary 6.12 (Auditable finite-N bound for g_M sampling)**.** *Let* $\mathcal{P} \in \mathsf{Scan}_{\mathrm{alg}}$ *be a* d*-dimensional protocol with regularized kernel* $f_{\mathcal{R}} = g_M$*. Then for the orbit prefix point set* P_N*,*

$$\left| \langle \mathcal{P} \rangle_N - H_M^{(d)} \right| \leq (2^d - 1)(M - 1)\, D_N^*(P_N). \tag{39}$$

Proof. Combine Corollary 6.3 with Proposition 6.11. □

For $d = 2$ and $d = 3$, the limits $H_M^{(2)} \to \zeta(2)$ and $H_M^{(3)} \to \zeta(3)$ are classical, and the integral test yields explicit tails.

Proposition 6.13 (Truncation bounds for $\zeta(2)$ and $\zeta(3)$)**.** *For $M \geq 1$,*

$$0 < \zeta(2) - H_M^{(2)} < \frac{1}{M}, \tag{40}$$

$$0 < \zeta(3) - H_M^{(3)} < \frac{1}{2M^2}. \tag{41}$$

Proof. Use the integral remainder bound for decreasing positive series:

$$\sum_{n=M+1}^{\infty} \frac{1}{n^s} \leq \int_M^{\infty} x^{-s} \mathrm{d}x = \frac{1}{(s-1)M^{s-1}}, \qquad s > 1,$$

and take $s = 2$ and $s = 3$. □

Remark 6.14 (Why truncation is preferable for auditability)**.** *The "ideal" kernel $1/(1-xy)$ on $[0,1]^2$ is not bounded near $xy = 1$, complicating finite-N sampling bounds. By replacing it with the polynomial kernel $g_M(x,y) = \sum_{n=0}^{M-1}(xy)^n$, we obtain: (i) an exact period $H_M^{(2)}$ at finite M, (ii) a uniform integrand regularity compatible with discrepancy bounds, and (iii) an explicit truncation remainder controlled by* (40)*. The same strategy applies to $d = 3$.*

Corollary 6.15 (Stability–truncation tradeoff and an optimized choice of M for $\zeta(d)$)**.** *Fix an integer $d \geq 2$ and let g_M be the truncated geometric kernel (Definition 6.9) on $[0,1)^d$. Let $\mathcal{P} \in \mathsf{Scan}_{\mathrm{alg}}$ be a d-dimensional protocol with regularized kernel $f_{\mathcal{R}} = g_M$ and orbit-prefix point set P_N. Assume a (possibly protocol-dependent) explicit discrepancy certificate*

$$D_N^*(P_N) \leq \delta_N.$$

Then the total deviation from $\zeta(d)$ satisfies the auditable bound

$$|\langle \mathcal{P} \rangle_N - \zeta(d)| \leq \underbrace{(2^d - 1)(M-1)\,\delta_N}_{\textit{sampling error}} + \underbrace{\frac{1}{(d-1)M^{d-1}}}_{\textit{truncation error}}. \tag{42}$$

In particular, with $A := (2^d - 1)\delta_N$ and the integer choice

$$M := \left\lceil A^{-1/d} \right\rceil, \tag{43}$$

one obtains the explicit rate bound

$$|\langle \mathcal{P} \rangle_N - \zeta(d)| \leq \frac{d}{d-1} A^{(d-1)/d} + A. \tag{44}$$

For $d = 2$ this yields $|\langle \mathcal{P} \rangle_N - \zeta(2)| \leq 2\sqrt{3\delta_N} + 3\delta_N$, and for $d = 3$ it yields $|\langle \mathcal{P} \rangle_N - \zeta(3)| \leq \frac{3}{2}(7\delta_N)^{2/3} + 7\delta_N$.

Proof. By Corollary 6.12 and Lemma 6.10,

$$\left| \langle \mathcal{P} \rangle_N - H_M^{(d)} \right| \leq (2^d - 1)(M-1)\, D_N^*(P_N) \leq (2^d - 1)(M-1)\,\delta_N.$$

For the truncation remainder, the integral test gives

$$0 < \zeta(d) - H_M^{(d)} = \sum_{n=M+1}^{\infty} \frac{1}{n^d} \leq \int_M^{\infty} x^{-d}\, \mathrm{d}x = \frac{1}{(d-1)M^{d-1}},$$

which yields (42) by Proposition 6.7. For (44), note that $M \geq A^{-1/d}$ implies $M^{-(d-1)} \leq A^{(d-1)/d}$ and that $M - 1 \leq M \leq A^{-1/d} + 1$, hence

$$(2^d - 1)(M-1)\delta_N \leq A\left(A^{-1/d} + 1\right) = A^{(d-1)/d} + A, \qquad \frac{1}{(d-1)M^{d-1}} \leq \frac{1}{d-1} A^{(d-1)/d}.$$

Summing these bounds gives (44). □

6.5 Benchmark one-dimensional kernels: explicit variation constants

For monotone functions $f : [0,1] \to \mathbb{R}$, the Jordan variation satisfies $\mathrm{Var}(f) = |f(1) - f(0)|$. Therefore the $d = 1$ Koksma inequality (the $d = 1$ case of Theorem 6.1) yields explicit constants for the benchmark kernels used in Section 9.

Proposition 6.16 (Explicit 1D variation constants)**.** *Let* $f_{\log}(x) = \frac{1}{1+x}$ *and* $f_\pi(x) = \frac{4}{1+x^2}$ *on* $[0,1]$*. Then*

$$\mathrm{Var}(f_{\log}) = \frac{1}{2}, \qquad \mathrm{Var}(f_\pi) = 2. \tag{45}$$

Proof. Both kernels are smooth and strictly decreasing on $[0,1]$. Hence $\mathrm{Var}(f) = f(0) - f(1)$. □

Proposition 6.17 (Hardy–Krause variation for separable products)**.** *Let* $f_j : [0,1] \to \mathbb{R}$ *be* C^1 *functions with finite Jordan variation* $\mathrm{Var}(f_j) < \infty$*, and define*

$$F(x_1, \dots, x_d) := \prod_{j=1}^{d} f_j(x_j) \qquad \text{on } [0,1]^d.$$

Then F *has finite Hardy–Krause variation (anchored at* 1*) and*

$$\mathrm{Var}_{\mathrm{HK}}(F) = \prod_{j=1}^{d} \left(\mathrm{Var}(f_j) + |f_j(1)|\right) \; - \; \prod_{j=1}^{d} |f_j(1)|. \tag{46}$$

Proof. For C^1 functions, the anchored Hardy–Krause variation admits the mixed-derivative representation (Remark B.4). For a nonempty subset $u \subset \{1, \dots, d\}$, the mixed derivative of $F(x_u; 1)$ factorizes:

$$\frac{\partial^{|u|}}{\prod_{j \in u} \partial x_j} F(x_u; 1) = \left(\prod_{j \in u} f_j'(x_j) \right) \left(\prod_{k \notin u} f_k(1) \right).$$

Therefore

$$\int_{[0,1]^{|u|}} \left| \frac{\partial^{|u|}}{\prod_{j \in u} \partial x_j} F(x_u; 1) \right| \mathrm{d}x_u = \left(\prod_{k \notin u} |f_k(1)| \right) \left(\prod_{j \in u} \int_0^1 |f_j'(x)| \, \mathrm{d}x \right) = \left(\prod_{k \notin u} |f_k(1)| \right) \left(\prod_{j \in u} \mathrm{Var}(f_j) \right),$$

since f_j is C^1 and $\mathrm{Var}(f_j) = \int_0^1 |f_j'(x)| \, \mathrm{d}x$. Summing over all nonempty u and expanding the product gives (46). □

Corollary 6.18 (Explicit Hardy–Krause constants for the π^2 and π^3 kernels)**.** *Let* $f_\pi(x) = \frac{4}{1+x^2}$ *on* $[0,1]$ *and define*

$$F_2(x, y) := \frac{16}{(1+x^2)(1+y^2)} = f_\pi(x) f_\pi(y), \qquad F_3(x, y, z) := \frac{64}{(1+x^2)(1+y^2)(1+z^2)} = f_\pi(x) f_\pi(y) f_\pi(z).$$

Then

$$\mathrm{Var}_{\mathrm{HK}}(F_2) = 12, \qquad \mathrm{Var}_{\mathrm{HK}}(F_3) = 56. \tag{47}$$

Proof. By Proposition 6.16, $\mathrm{Var}(f_\pi) = 2$ and $f_\pi(1) = 2$. Apply Proposition 6.17 with $d = 2$ and $d = 3$. □

6.6 Propagation rules for composed protocols

Auditability requires that error budgets compose. For the present paper, it suffices to record the following simple propagation rule.

Lemma 6.19 (Linear combination rule)**.** *Let $\widehat{P}_i$ be estimators for numbers P_i with bounds $|\widehat{P}_i - P_i| \le \varepsilon_i$. For any scalars $c_i \in \mathbb{R}$,*

$$\left|\sum_i c_i \widehat{P}_i - \sum_i c_i P_i\right| \le \sum_i |c_i| \varepsilon_i. \tag{48}$$

This rule is sufficient for the low-complexity constant search of Section 9.6, where candidates are linear combinations of precomputed period primitives.

7 A selection principle: stability under finite resources

The realization theorem shows that, under equidistribution, the asymptotic value of a scan protocol depends only on the period datum. In particular, for a fixed kernel $f_{\mathcal{R}}$, the integral is independent of the scan slope α as long as (21) holds. This shifts the locus of "selection" away from the infinite-time limit and toward the finite-resource regime: what differs across protocols is not the limit value but the *stability and cost* with which that value is realized.

7.1 Value–rate separation

Let $\mathcal{P} = (\mathbb{T}^d, \alpha, x_0, f, \mathcal{R})$ and $\mathcal{P}' = (\mathbb{T}^d, \alpha', x_0', f, \mathcal{R})$ share the same kernel but use different scan slopes. If both slopes satisfy (21), then

$$\lim_{N\to\infty} \langle \mathcal{P} \rangle_N = \lim_{N\to\infty} \langle \mathcal{P}' \rangle_N = \int_{[0,1)^d} f_{\mathcal{R}}.$$

However, the finite-N errors generally differ:

$$\left|\langle \mathcal{P} \rangle_N - \int f_{\mathcal{R}}\right| \text{ depends on } \alpha \quad \text{through} \quad D_N^*(P_N),$$

where $P_N = \{x_0, \ldots, x_{N-1}\}$ denotes the scan-orbit prefix on $[0,1)^d$. This is the operational content of value–rate separation: selection does not choose the integral, but chooses how efficiently the protocol realizes it.

7.2 Stability metrics and bounded description budgets

To make the notion of selection falsifiable, we require a cost functional that can be evaluated (or upper-bounded) from auditable protocol data.

Definition 7.1 (Finite-horizon stability bound)**.** *For a protocol $\mathcal{P} \in \mathsf{Scan}_{\mathrm{alg}}$ at horizon N, define a stability bound*

$$\mathrm{Stab}_N(\mathcal{P}) := \mathrm{Var}_{\mathrm{HK}}(f_{\mathcal{R}})\, D_N^*(P_N) \tag{49}$$

whenever $\mathrm{Var}_{\mathrm{HK}}(f_{\mathcal{R}}) < \infty$*.*

Definition 7.1 uses Corollary 6.3 as a canonical, auditable choice. Alternative bounds (e.g. bounds in terms of continued-fraction digits in $d = 1$) may be substituted, provided they remain auditable within the protocol.

Selection also requires a cost for *describing* the protocol. In the present controlled setting, the dominant contributions are: the symbolic complexity of the kernel (degree, coefficient sizes, number of terms), and the regularization parameters (e.g. the truncation depth M).

Definition 7.2 (Protocol description complexity (schematic)). *Let* $\mathrm{Comp}(\mathcal{P}; N)$ *denote a computable surrogate for the description-and-implementation cost of running a protocol* $\mathcal{P}$ *up to horizon* N*. A concrete default choice in* $\mathsf{Scan}_{\mathrm{alg}}$ *is*

$$\mathrm{Comp}(\mathcal{P}; N) := d \;+\; \mathrm{size}(f_{\mathcal{R}}) \;+\; \mathrm{cost}(\mathcal{R}) \;+\; \log(1+N), \tag{50}$$

where $\mathrm{size}(f_{\mathcal{R}})$ *is a syntactic size of a rational presentation of* $f_{\mathcal{R}}$ *(e.g. total number of nonzero coefficients plus degree and coefficient height), and* $\mathrm{cost}(\mathcal{R})$ *accounts for regularization parameters (e.g. truncation depth* M*). Different reasonable encodings change* (50) *by at most multiplicative/additive constants (in the sense of invariance theorems for description complexity) and are interchangeable at the level of the programmatic conjecture; see, e.g., [16].*

Definition 7.3 (Naive height of period data). *Let* $f_{\mathcal{R}}$ *be represented as a reduced rational function* p/q *with* $p, q \in \mathbb{Z}[x_1, \dots, x_d]$ *having no common factor. Define the naive height*

$$H(f_{\mathcal{R}}) := \max\{H(p), H(q)\}, \qquad H(p) := \max\{|c| : \ c \text{ is a coefficient of } p\}. \tag{51}$$

Any rational presentation in $\mathbb{Q}(x_1, \dots, x_d)$ *can be put into this form by clearing denominators and dividing by the content* gcd*; the resulting height is well-defined up to the obvious sign normalization. For a cubical period datum* $\mathrm{HSP}(\mathcal{P}) = ([0,1)^d, f, \mathcal{R})$*, define*

$$\mathrm{Height}(\mathrm{HSP}(\mathcal{P})) := \log\left(1 + H(f_{\mathcal{R}})\right) + \deg(p) + \deg(q), \tag{52}$$

as a computable arithmetic-geometric size surrogate.

Remark 7.4 (Example: truncated geometric kernels). *For* $g_M(x) = \sum_{n=0}^{M-1} (\prod_{j=1}^{d} x_j)^n$*, one has* $H(g_M) = 1$ *and* $\mathrm{cost}(\mathcal{R})$ *is naturally linear in* M *if* g_M *is evaluated as a length-*M *sum, or logarithmic in* M *if evaluated via the closed form* $(1 - p^M)/(1 - p)$ *with fast exponentiation. In either case, the truncation depth is a direct, auditable complexity knob in* (50)*.*

7.3 Selection principle (programmatic conjecture)

We now state a programmatic, falsifiable selection principle. It is not used as an input to any closed theorem.

Conjecture 7.5 (Selection by stability–complexity optimization). *There exists a functional* $\mathcal{J}_N$ *on a class of admissible protocols* $\mathcal{C} \subset \mathsf{Scan}_{\mathrm{alg}}$ *of the form*

$$\mathcal{J}_N(\mathcal{P}) = \mathrm{Stab}_N(\mathcal{P}) \;+\; \lambda\, \mathrm{Comp}(\mathcal{P}; N) \;+\; \mu\, \mathrm{Height}(\mathrm{HSP}(\mathcal{P})), \tag{53}$$

with $\lambda, \mu \geq 0$*, such that the period data empirically observed as "constants" correspond to near-minimizers (or extremizers) of* $\mathcal{J}_N$ *as* N *ranges over accessible horizons.*

The term $\mathrm{Height}(\mathrm{HSP}(\mathcal{P}))$ is optional and programmatic: it is intended to encode arithmetic/geometric "size" of the period datum (in the spirit of heights in arithmetic geometry) and to penalize excessively high-complexity realizations. The conjecture is falsifiable because each term in (53) can be operationalized: Stab_N can be upper-bounded by discrepancy bounds, Comp is a bounded description budget, and candidate height surrogates can be compared by predictive performance.

7.4 An operational falsification protocol (schematic)

To make the term "falsifiable" concrete in the present audit setting, one may fix:

- a finite resource horizon N and a finite description budget (which determines a finite candidate subclass $\mathcal{C}_{N,B} \subset \mathsf{Scan}_{\mathrm{alg}}$),

- a concrete encoding of Comp (Definition 7.2) and a concrete height surrogate (Definition 7.3),
- weights λ, μ (which may be fit on a calibration set and then frozen).

For each candidate protocol $\mathcal{P} \in \mathcal{C}_{N,B}$ one then computes (or upper-bounds) $\mathcal{J}_N(\mathcal{P})$ from auditable data. Given an empirical constant target T with an uncertainty interval, one tests whether there exists a near-minimizer family $\{\mathcal{P}_\star\}$ whose realized period values fall inside that interval while maintaining low $\mathcal{J}_N$. If, across a sufficiently rich and auditable candidate class, observed constants systematically fail to align with near-minimizers (or if low-$\mathcal{J}_N$ near-minimizers systematically predict values not observed), the conjecture is falsified in that regime. Lemma 7.7 isolates a quantitative rigidity mechanism that makes such tests robust to finite uncertainty bars.

7.5 Why the golden branch appears

In one dimension, the discrepancy of Kronecker sequences is controlled by Diophantine properties of α, typically expressed via continued fractions [2, 17]. Badly approximable slopes (bounded continued-fraction coefficients) yield improved uniformity bounds at finite horizons. The golden ratio φ is extremal in the sense that it is the *most* badly approximable number: it maximizes the constant $c(\alpha)$ in inequalities of the form

$$\left|\alpha - \frac{p}{q}\right| \geq \frac{c(\alpha)}{q^2},$$

over irrational α.

More precisely, Hurwitz's theorem implies that for every irrational α there exist infinitely many rationals p/q with

$$\left|\alpha - \frac{p}{q}\right| < \frac{1}{\sqrt{5}\,q^2},$$

and the constant $1/\sqrt{5}$ is best possible; the extremal case is attained on the golden branch (continued fraction $[1; 1, 1, \dots]$) and its $\mathrm{SL}_2(\mathbb{Z})$ orbit [17, 18].

This extremality can be upgraded from a qualitative heuristic to an explicit stability certificate in our audit framework. If α has bounded partial quotients $a_n \leq A$, then Proposition B.1 yields the closed bound

$$D_N^*(P_N) \leq \frac{2A(3 + \log_\varphi N)}{N}.$$

For a one-dimensional protocol with bounded-variation kernel $f_\mathcal{R}$, the $d = 1$ Koksma inequality therefore gives the explicit finite-resource guarantee

$$|\langle \mathcal{P} \rangle_N - \mathrm{per}(\mathrm{HSP}(\mathcal{P}))| \leq \mathrm{Var}(f_\mathcal{R})\, D_N^*(P_N) \leq \mathrm{Var}(f_\mathcal{R})\, \frac{2A(3 + \log_\varphi N)}{N}.$$

The golden branch has $A = 1$, minimizing this constant-type certificate.

Within the selection principle framework, the golden branch is therefore not a numerological artifact: it is a concrete scan choice that improves stability bounds for a wide class of kernels under finite resources.

7.6 Protocol-stable period data as stable period structures

Definition 3.7 can now be read as a stability constraint: a protocol-stable period datum is a period datum that admits realizations whose $\mathcal{J}_N$ cost remains small under bounded resources, and whose value is invariant under protocol equivalence. Under this perspective, "motive" provides a language for organizing relations among such stable period data, while selection is enforced at the protocol layer through finite-horizon stability and complexity.

7.7 Dimension cost and an effective curse of dimensionality

The selection principle becomes sharper when dimension is made explicit. For explicit low-discrepancy constructions in quasi-Monte Carlo integration, one has upper bounds of the form

$$D_N^* \le C_d \, \frac{(\log N)^{d-1}}{N}, \tag{54}$$

for some dimension-dependent constant C_d for explicit low-discrepancy constructions [3,4]. Combined with Koksma–Hlawka, this yields the generic finite-horizon estimate

$$\Big| \langle \mathcal{P} \rangle_N - \mathrm{per}(\mathrm{HSP}(\mathcal{P})) \Big| \le \mathrm{Var}_{\mathrm{HK}}(f_{\mathcal{R}}) \, C_d \, \frac{(\log N)^{d-1}}{N}. \tag{55}$$

Thus, for fixed horizon N, the auditable bound deteriorates rapidly with dimension d; equivalently, achieving a target accuracy ε requires a horizon satisfying

$$N \ge \frac{\mathrm{Var}_{\mathrm{HK}}(f_{\mathcal{R}}) \, C_d}{\varepsilon} \, (\log N)^{d-1}. \tag{56}$$

In the present program, this motivates an explicit *dimension penalty* in the selection functional (53): high-dimensional period data are "expensive" to stabilize under bounded resources, and low-dimensional periods (1D/2D/3D) are naturally favored as stable observables.

Remark 7.6 (Lower bounds: the dimension penalty is not removable)**.** *The deterioration with d is not merely an artifact of a particular construction. For $d \ge 2$, Roth's theorem gives a general lower bound on star discrepancy: for any N-point set in $[0,1]^d$, one has*

$$D_N^* \ge c_d \, \frac{(\log N)^{(d-1)/2}}{N}$$

for a positive constant c_d depending only on d [19, 20]. In particular, in dimension $d = 2$ one can strengthen the logarithmic exponent [21]. These bounds support the interpretation that high-dimensional stable realizations are intrinsically more expensive under finite resources.

7.8 Robustness under target perturbations (rigidity via gap)

Many "selection" statements are metrological in nature: one compares a finite candidate class against a target value known only up to an error bar. The following lemma isolates a purely quantitative rigidity mechanism: a large best-vs-second-best gap implies robustness under perturbations.

Lemma 7.7 (Gap-stability of a unique minimizer)**.** *Let V be a finite set of real candidates and let $T \in \mathbb{R}$ be a target. Let $v_\star \in V$ be the unique minimizer of $|v - T|$, and let $v_2 \in V$ be a minimizer among $V \setminus \{v_\star\}$. Define the margin*

$$m := |v_2 - T| - |v_\star - T| > 0. \tag{57}$$

Then for every perturbed target $T' \in \mathbb{R}$ with $|T' - T| < m/2$, the unique minimizer of $|v - T'|$ over $v \in V$ is still $v_\star$.

Proof. For any $v \in V$, triangle inequality gives

$$|v - T'| \ge |v - T| - |T' - T|, \qquad |v_\star - T'| \le |v_\star - T| + |T' - T|.$$

Hence for $v \ne v_\star$,

$$|v - T'| - |v_\star - T'| \ge \big(|v - T| - |v_\star - T|\big) - 2|T' - T| \ge m - 2|T' - T| > 0.$$

□

8 Compatibility target: factoring the Langlands upgrade through periods and motives

The earlier "stairway" work [1] emphasized a functorial upgrade objective: to construct a holographic Langlands functor from a category of scan protocols into a category of automorphic representations, with provable compatibility across renormalization flow, cusp discretization, and Hecke dynamics. The present paper contributes a new, auditable arrow—the functor $\mathrm{HSP} : \mathsf{Scan}_{\mathrm{alg}} \to \mathsf{PerDatum}$—and uses it to formulate a refined compatibility target.

8.1 A commutative-diagram objective

In the classical modular setting, three layers are tightly related:

- **Period layer**: integrals of algebraic or modular differential forms against cycles, yielding periods (Section 3.1).
- **ℓ-adic layer**: Galois representations whose Frobenius traces encode arithmetic spectra, such as Hecke eigenvalues in the modular case [22, 23].
- **Automorphic layer**: automorphic representations organizing Hecke actions and their local parameters [24].

Motives provide a conjecturally universal organization principle for these compatibilities, with periods appearing as comparison invariants between realizations.

The present work motivates the following objective for the audit program:

$$\mathsf{Scan}_{\mathrm{alg}} \xrightarrow{\mathrm{HSP}} \mathsf{PerDatum} \xrightarrow{\mathrm{Mot}} \mathsf{Mot} \xrightarrow{\mathrm{Real}_\ell} \mathsf{GalRep} \qquad \text{and} \qquad \mathsf{Mot} \xrightarrow{\mathrm{Aut}} \mathsf{AutRep}, \tag{58}$$

where:

- Mot is a suitable category of motives (or a computable subcategory, e.g. mixed Tate motives),
- Mot attaches a motive to a period datum when available (programmatic),
- Real_ℓ takes an ℓ-adic realization, producing a Galois representation,
- Aut associates an automorphic representation (or parameter) in regimes where Langlands correspondences are known.

In this diagram, the arrow HSP is *closed* and auditable within this paper. The arrows Mot, Real_ℓ, and Aut are classical in several standard regimes (e.g. mixed Tate motives and modular motives), but they are not available as a uniform construction on arbitrary period data. In the present paper we therefore treat them as programmatic interfaces and aim to instantiate them only in controlled test cases.

Appendix D records two such classical test cases. The first is the mixed Tate period calculus on $\mathbb{P}^1 \setminus \{0, 1, \infty\}$, where iterated-integral periods include $\log 2$ and $\zeta(2)$. The second is the modular/elliptic setting, where weight-2 newforms have period integrals controlling $L(f, 1)$ and admit ℓ-adic realizations with Frobenius traces given by Hecke eigenvalues.

8.2 How this relates to the stairway chain

The stairway chain [1] produces, within Layer 0/1, a route from scan protocols to the Hecke prime skeleton and cusp coefficient data. In standard arithmetic geometry, the same coefficient data can be interpreted as part of the realization data of modular motives: Hecke eigenvalues correspond to Frobenius traces in suitable ℓ-adic realizations, while periods arise from comparison between Betti and de Rham realizations.

The contribution of the present paper is to isolate a stable numerical interface in the middle: period data that are directly computable from scan averages. This isolates an auditable passage

$$\text{protocol} \;\Rightarrow\; \text{period datum} \;\Rightarrow\; (\text{candidate motive source}),$$

and clarifies the division of labor:

- the *closed* part explains why periods (hence $\pi, \log, \zeta$-values) emerge from protocols and how finite-resource errors are controlled;
- the *programmatic* part aims to integrate this period interface with the Hecke/Langlands interface of [1] to obtain a genuinely functorial closure.

8.3 A controlled modular-motive test case

As a concrete target for future work, one may restrict to modular forms of low weight/level and the associated modular motives. In this regime, both sides admit explicit computations: cusp forms have Fourier expansions with Hecke eigenvalues; their periods can be computed as integrals of modular forms on suitable cycles; and Deligne's construction attaches ℓ-adic Galois representations with matching traces [22, 23]. Appendix D spells out this classical chain at the level of period integrals and Frobenius traces.

The scan-period interface of this paper suggests a computable experiment: choose kernels whose associated period data are known to coincide with modular periods, and compare their scan realizations with the cusp/Hecke data extracted by the stairway pipeline. Such a comparison would provide an auditable bridge from scan protocols to motivic realizations in a controlled modular setting.

9 Reproducible experiments (pure Python)

This section provides reproducible toy experiments that validate the paper's *new computable claims*: scan protocols in $\mathsf{Scan}_{\mathrm{alg}}$ realize periods and admit auditable error decompositions. Standard mathematical facts (equidistribution, period closure properties) are not "proven" numerically; the experiments only demonstrate that the proposed implementations behave as predicted under finite resources.

9.1 Reproducibility protocol

All experiments are implemented in pure Python 3 with no third-party dependencies. To reproduce the results:

- run `python3 scripts/period_scan_experiments.py` to generate the scan-period tables;
- run `python3 scripts/alpha_inverse_search.py` to generate the low-complexity constant search table.

The scripts write LaTeX row files into `sections/generated/` which are included below.

9.2 Experiment I: $\log 2$ via a one-dimensional scan

We use the golden-branch slope $\alpha = \varphi^{-1}$ and the kernel $f(x) = \frac{1}{1+x}$ (Example 5.6). The target value is $\log 2$.

N	$\langle \mathcal{P} \rangle_N$	$\langle \mathcal{P} \rangle_N - \log 2$	D_N^*	$\mathrm{Var}(f)\, D_N^*$	ratio
10,000	0.693107871664	-3.93×10^{-5}	3×10^{-4}	1.5×10^{-4}	2.6×10^{-1}
50,000	0.693147942126	$+7.62 \times 10^{-7}$	5.1×10^{-5}	2.6×10^{-5}	3×10^{-2}
200,000	0.693147233266	$+5.27 \times 10^{-8}$	1.2×10^{-5}	6.1×10^{-6}	8.7×10^{-3}

Table 1: $\log 2$ realized as a scan period via the kernel $f(x) = 1/(1+x)$ and the golden-branch slope $\alpha = \varphi^{-1}$. The scripts compute the one-dimensional star discrepancy D_N^* exactly from its definition and report the deterministic Koksma certificate $|\langle \mathcal{P} \rangle_N - \log 2| \le \mathrm{Var}(f)\, D_N^*$ (Proposition 6.16).

N	$\langle \mathcal{P} \rangle_N$	$\langle \mathcal{P} \rangle_N - \pi$	D_N^*	$\mathrm{Var}(f)\, D_N^*$	ratio
10,000	3.141419572078	-1.73×10^{-4}	3×10^{-4}	6×10^{-4}	2.9×10^{-1}
50,000	3.141596841077	$+4.19 \times 10^{-6}$	5.1×10^{-5}	1×10^{-4}	4.1×10^{-2}
200,000	3.141593512097	$+8.59 \times 10^{-7}$	1.2×10^{-5}	2.4×10^{-5}	3.5×10^{-2}

Table 2: π realized as a scan period via the kernel $f(x) = 4/(1+x^2)$ and the golden-branch slope. As in Table 1, the scripts compute D_N^* exactly and report the deterministic Koksma certificate.

A closed, parameter-free finite-N certificate for the golden branch. Since $\alpha = \varphi^{-1}$ has bounded continued-fraction coefficients, Proposition B.1 yields the explicit bound

$$D_N^*(P_N) \le \frac{2(3 + \log_\varphi N)}{N}.$$

Combining this with Proposition 6.16 gives the fully explicit inequality

$$|\langle \mathcal{P} \rangle_N - \log 2| \le \mathrm{Var}(f_{\log})\, D_N^*(P_N) \le \frac{3 + \log_\varphi N}{N}, \qquad f_{\log}(x) = \frac{1}{1+x},$$

independent of any numerical estimation of discrepancy.

A simple rate fit (log–log). Using the same data, the script also performs a least-squares fit of $\log |\langle \mathcal{P} \rangle_N - \log 2|$ against $\log N$ over the tabulated horizons, yielding the empirical slope

$$\beta_{\log 2} = -2.214,$$

which should be read as a descriptive summary of these horizons rather than a claimed asymptotic law. The closed certificate above is $O((\log N)/N)$, while the observed absolute error can decay faster on specific kernels and finite ranges.

9.3 Experiment II: π via a one-dimensional scan

We keep the same slope $\alpha = \varphi^{-1}$ and use the kernel $f(x) = \frac{4}{1+x^2}$ (Example 5.7). The target value is π.

A closed, parameter-free finite-N certificate for the golden branch. Combining Proposition B.1 with Proposition 6.16 yields

$$|\langle \mathcal{P} \rangle_N - \pi| \le \mathrm{Var}(f_\pi)\, D_N^*(P_N) \le \frac{4(3 + \log_\varphi N)}{N}, \qquad f_\pi(x) = \frac{4}{1+x^2}.$$

A simple rate fit (log–log). Analogously, the least-squares log–log slope for $|\langle \mathcal{P} \rangle_N - \pi|$ over the tabulated horizons is

$$\beta_\pi = -1.786.$$

(N,M,K)	H	$\overline{\langle\mathcal{P}\rangle}_{N_K}$	$\overline{\langle\mathcal{P}\rangle}_{N_K}-\zeta(2)$	$\overline{\langle\mathcal{P}\rangle}_{N_K}-H_M^{(2)}$	trunc. bound $<1/M$	$B_{N,H}$	$\frac{9}{4}\,\mathrm{Var}_{\mathrm{HK}}(g_M)\,B_{N,H}$	ratio
(200,000,5,000,10)	300	1.644033442945	-9.01×10^{-4}	-7.01×10^{-4}	2×10^{-4}	1.3×10^{-2}	4.4×10^{2}	1.6×10^{-6}
(1,000,000,5,000,10)	800	1.644335771403	-5.98×10^{-4}	-3.98×10^{-4}	2×10^{-4}	3.9×10^{-3}	1.3×10^{2}	3×10^{-6}
(2,000,000,20,000,20)	800	1.644862011186	-7.21×10^{-5}	-2.21×10^{-5}	5×10^{-5}	2.6×10^{-3}	3.5×10^{2}	6.3×10^{-8}

Table 3: $\zeta(2)$ realized via a two-dimensional scan and an auditable truncation. The total error is decomposed into sampling error (fifth column) and truncation error (sixth column bound). For each (N,M,K) case, the ETK truncation parameter H (second column) is chosen by minimizing the computable Kronecker bracket term $B_{N,H}(\alpha;N)$ (Definition B.11) over the fixed candidate set $\mathcal{H}_2=\{10,20,30,40,50,80,100,150,200,300,500,800\}$. The last three columns report this $B_{N,H}(\alpha;N)$ (written as $B_{N,H}$ in the table) and the resulting fully numerical sampling certificate obtained from the ETK inequality with the explicit constant $C_2=(3/2)^2=9/4$ (Theorem B.5): namely $D_N^*(P_N)\le\frac{9}{4}B_{N,H}(\alpha;N)$ and hence $|\overline{\langle\mathcal{P}\rangle}_{N_K}-H_M^{(2)}|\le\frac{9}{4}\mathrm{Var}_{\mathrm{HK}}(g_M)B_{N,H}(\alpha;N)$. The ratio column is $|\overline{\langle\mathcal{P}\rangle}_{N_K}-H_M^{(2)}|/(\frac{9}{4}\mathrm{Var}_{\mathrm{HK}}(g_M)B_{N,H}(\alpha;N))$.

9.4 Experiment III: $\zeta(2)$ via a two-dimensional scan with truncation

We use the truncated geometric kernel $g_M(x,y)=\sum_{n=0}^{M-1}(xy)^n$ (Definition 6.9), whose exact period is $H_M^{(2)}$ (Lemma 6.10). The truncation remainder satisfies $\zeta(2)-H_M^{(2)}<1/M$ (Proposition 6.13). The scan slope is $\alpha=(\varphi^{-1},\sqrt{2}-1)$ and we report an ensemble mean over K random initial conditions (Definition 4.8) with a fixed seed.

A closed total-error certificate. For each (N,M,K) case, Proposition 6.7 and Corollary 6.3 yield the deterministic inequality

$$\left|\overline{\langle\mathcal{P}\rangle}_{N_K}-\zeta(2)\right|\le\left|\overline{\langle\mathcal{P}\rangle}_{N_K}-H_M^{(2)}\right|\;+\;\left(\zeta(2)-H_M^{(2)}\right)\le\mathrm{Var}_{\mathrm{HK}}(g_M)\,D_N^*(P_N)\;+\;\frac{1}{M},$$

where the right-hand side is auditable from protocol data once a discrepancy certificate for P_N is fixed. Corollary 6.15 makes the stability–truncation tradeoff explicit and provides an optimized choice of M once an explicit upper bound on $D_N^*(P_N)$ is available.

On conservatism of the multi-dimensional certificates. For $d\ge2$ we report the computable bracket term $B_{N,H}(\alpha;N)$ (Definition B.11) together with the explicit choice $C_d=(3/2)^d$ in Theorem B.5. The truncation parameter H is chosen from a fixed candidate list by minimizing $B_{N,H}(\alpha;N)$ (reported in the tables). These choices are made for audit simplicity and uniform reproducibility rather than tightness; the bound can be tightened further by enlarging the candidate list, using sharper variants of ETK, or using sharper Diophantine information on α. The closed theoretical conclusions of the paper do not depend on any particular constant choice.

9.5 Experiment IV: $\zeta(3)$ via a three-dimensional scan with truncation

We use the kernel $g_M(x,y,z)=\sum_{n=0}^{M-1}(xyz)^n$ with exact period $H_M^{(3)}$ and truncation remainder $\zeta(3)-H_M^{(3)}<1/(2M^2)$ (Proposition 6.13). The scan slope is $\alpha=(\varphi^{-1},\sqrt{2}-1,\sqrt{3}-1)$.

A certified numerical reference for $\zeta(3)$. Since $\zeta(3)$ has no known closed form, we use a certified reference interval obtained from the defining series. Fix $M_{\mathrm{ref}}=200{,}000$ and set $H_{M_{\mathrm{ref}}}^{(3)}=\sum_{n=1}^{M_{\mathrm{ref}}}n^{-3}$. Proposition 6.13 implies

$$\zeta(3)\in\left[H_{M_{\mathrm{ref}}}^{(3)},\,H_{M_{\mathrm{ref}}}^{(3)}+\frac{1}{2M_{\mathrm{ref}}^2}\right].$$

We define $\zeta(3)_{\mathrm{ref}}$ as the midpoint of this certified interval, so that $|\zeta(3)-\zeta(3)_{\mathrm{ref}}|\le1/(4M_{\mathrm{ref}}^2)$.

(N, M, K)	H	$\overline{\langle \mathcal{P} \rangle}_{N\,K}$	$\overline{\langle \mathcal{P} \rangle}_{N\,K} - \zeta(3)_{\rm ref}$	$\overline{\langle \mathcal{P} \rangle}_{N\,K} - H_M^{(3)}$	trunc. bound $< 1/(2M^2)$	$B_{N,H}$	$\frac{27}{8}\,\mathrm{Var}_{\rm HK}(g_M)\,B_{N,H}$	ratio
(200,000,2,000,10)	30	1.202133251644	$+7.63 \times 10^{-5}$	$+7.65 \times 10^{-5}$	1.2×10^{-7}	7.3×10^{-2}	3.4×10^{3}	2.2×10^{-8}
(1,000,000,2,000,10)	100	1.202099490850	$+4.26 \times 10^{-5}$	$+4.27 \times 10^{-5}$	1.2×10^{-7}	2.7×10^{-2}	1.3×10^{3}	3.3×10^{-8}

Table 4: $\zeta(3)$ realized via a three-dimensional scan with auditable truncation. The third column uses the certified reference $\zeta(3)_{\rm ref}$ defined above; it satisfies $|\zeta(3) - \zeta(3)_{\rm ref}| \le 1/(4M_{\rm ref}^2)$. As in Table 3, for each (N, M, K) case the ETK truncation parameter H (second column) is chosen by minimizing the computable Kronecker bracket term $B_{N,H}(\alpha; N)$ (Definition B.11) over the fixed candidate set $\mathcal{H}_3 = \{10, 20, 30, 40, 50, 60, 80, 100\}$. The last three columns report this $B_{N,H}(\alpha; N)$ (written as $B_{N,H}$ in the table) and the resulting fully numerical sampling certificate obtained from the ETK inequality with the explicit constant $C_3 = (3/2)^3 = 27/8$ (Theorem B.5): namely $D_N^*(P_N) \le \frac{27}{8} B_{N,H}(\alpha; N)$ and hence $|\overline{\langle \mathcal{P} \rangle}_{N\,K} - H_M^{(3)}| \le \frac{27}{8}\mathrm{Var}_{\rm HK}(g_M) B_{N,H}(\alpha; N)$. The ratio column is $|\overline{\langle \mathcal{P} \rangle}_{N\,K} - H_M^{(3)}|/(\frac{27}{8}\mathrm{Var}_{\rm HK}(g_M) B_{N,H}(\alpha; N))$.

A closed total-error certificate. Independently of the reference $\zeta(3)_{\rm ref}$, Proposition 6.7 and Corollary 6.3 imply the deterministic bound

$$\left|\overline{\langle \mathcal{P} \rangle}_{N\,K} - \zeta(3)\right| \le \left|\overline{\langle \mathcal{P} \rangle}_{N\,K} - H_M^{(3)}\right| \;+\; \left(\zeta(3) - H_M^{(3)}\right) \le \mathrm{Var}_{\rm HK}(g_M)\, D_N^*(P_N) \;+\; \frac{1}{2M^2}.$$

As in the $d = 2$ case, Corollary 6.15 provides an optimized truncation choice once a discrepancy certificate is fixed.

9.6 Experiment V: low-complexity constant search for α^{-1}

As a toy model for "selection under a bounded description budget", we reproduce an exhaustive search over the low-complexity ansatz

$$v(a, b, c) = a\pi^3 + b\pi^2 + c\pi, \qquad a, b, c \in \mathbb{Z}_{\ge 0}, \quad a + b + c \le 10, \tag{59}$$

targeting the CODATA 2022 central value $\alpha^{-1}(0) = 137.035999177$ [25]. This complexity domain contains the geometric candidate $(a, b, c) = (4, 1, 1)$ used in the constants-geometry manuscript [26]. The goal here is not metrology, but a sharp *uniqueness gap* under a fixed budget.

Why this particular ansatz is auditable in the present framework. The primitives π, π^2, π^3 are themselves low-complexity cubical scan periods in $\mathsf{Scan}_{\rm alg}$ (Examples 5.7, 5.8, 5.9). Once a finite-resource certificate is available for each primitive at a chosen horizon, Lemma 6.19 shows that the propagated uncertainty of any linear combination is controlled by the coefficient ℓ^1 size $a + b + c$. This provides a direct stability rationale for the coefficient-sum budget and for restricting to the nonnegative cone (avoiding cancellation-driven instability).

Remark 9.1 (Running and scheme dependence (interpretation-layer note))**.** *The electromagnetic coupling $\alpha(\mu)$ is scale- and scheme-dependent in quantum field theory; CODATA quotes the low-energy value $\alpha(0)$ in a specific metrological convention. In this paper, the bounded-complexity search should be read as a* protocol-level *rigidity signal at a fixed reference convention: it tests whether a low-description geometric expression is uniquely selected within a constrained ansatz class. Any map from a geometric baseline to an operational coupling across scales belongs to interpretation-layer modeling and is treated elsewhere in the HPA–Ω program.*

Proposition 9.2 (Uniqueness gap at fixed coefficient-sum complexity)**.** *Within the complexity domain $a, b, c \in \mathbb{Z}_{\ge 0}$ and $a + b + c \le 10$, the unique minimizer of $|a\pi^3 + b\pi^2 + c\pi - \alpha^{-1}|$ (with $\alpha^{-1} = 137.035999177$) is $(a, b, c) = (4, 1, 1)$, with relative error 2.22×10^{-6}. The next-best triple in the same domain has relative error at least 3.24×10^{-3}.*

(a,b,c)	$a+b+c$	$v(a,b,c)$	Δ	Δ/α^{-1}
$(4,1,1)$	6	137.0363037759	$+3.05\times10^{-4}$	$+2.22\times10^{-6}$
$(4,0,4)$	8	136.5914773356	-4.45×10^{-1}	-3.24×10^{-3}
$(3,4,1)$	8	135.6388402988	-1.4×10^{0}	-1.02×10^{-2}
$(3,4,2)$	9	138.7804329524	$+1.74\times10^{0}$	$+1.27\times10^{-2}$
$(3,3,4)$	10	135.1940138585	-1.84×10^{0}	-1.34×10^{-2}

Table 5: Exhaustive search over $a\pi^3+b\pi^2+c\pi$ with $a,b,c\geq 0$ and $a+b+c\leq 10$, targeting $\alpha^{-1}=137.035999177$ (CODATA 2022 central) [25]. Here $\Delta=v(a,b,c)-\alpha^{-1}$. The minimizer is $(4,1,1)$ and the gap to the next best solution is large at fixed complexity.

Proof. This is a finite check by exhaustive enumeration over all triples $(a,b,c)\in\mathbb{Z}^3_{\geq 0}$ with $a+b+c\leq 10$ and minimization of the absolute error; see `scripts/alpha_inverse_search.py`, which generates the top-candidate table rows included in Table 5. ☐

Corollary 9.3 (Robustness under target perturbations). *Let $V=\{a\pi^3+b\pi^2+c\pi:\ a,b,c\in\mathbb{Z}_{\geq 0},\ a+b+c\leq 10\}$ and let $T=\alpha^{-1}=137.035999177$. Then the minimizer $v_\star=4\pi^3+\pi^2+\pi$ remains the unique minimizer of $|v-T'|$ over $v\in V$ for every perturbed target T' satisfying $|T'-T|<m/2$, where m is the best-vs-second-best margin in Lemma 7.7. In particular, the observed gap in Table 5 yields a wide robustness interval compared to metrological uncertainties.*

10 Conclusion

This paper introduced a period–motive interface as a minimal, audit-compatible "layer above the cusp" for the holographic scanning program. On a controlled protocol category $\mathsf{Scan}_{\mathrm{alg}}$ with algebraic (rational) readout kernels and explicit regularization rules, we constructed a holographic scanning functor HSP into a category of period data and proved a closed realization theorem: for rationally independent scan slopes, the long-time Birkhoff readout equals the Kontsevich–Zagier period associated with $\mathrm{HSP}(\mathcal{P})$.

For finite resources, we formulated an auditable error budget that decomposes the total deviation into (i) a discrepancy-controlled sampling error and (ii) a regularization/truncation error with provable bounds. Pure-Python experiments reproduced scan realizations of $\log 2$, π, $\zeta(2)$ and $\zeta(3)$ within this audited framework, and a bounded-complexity constant search illustrated a sharp uniqueness gap for a low-description ansatz targeting α^{-1}.

Beyond the closed chain, we stated a falsifiable selection principle: observable constants correspond to period data that are stably realizable under bounded resources and low description complexity. We also formulated a compatibility target that factors the functorial Langlands upgrade through period data and motives. In this sense, the present work does not replace the stairway; it supplies a source-level interface that explains why the climb repeatedly encounters the same special constants and provides a computable bridge to motivic organization.

A Audit status classification

This appendix records the audit status of the main components of the paper.

A.1 Closed chain (definitions and theorems)

- **Standard facts.**
  - Weyl/Kronecker equidistribution for irrational rotations (Theorems 5.1–5.3) [2, 13].
  - Discrepancy bounds such as Koksma–Hlawka (Theorem 6.1) [2].

- **New definitions.**
  - $\mathsf{Scan}_{\mathrm{alg}}$ (Definitions 4.1, 4.3).
  - $\mathsf{PerDatum}$ and period data (Definitions 3.3, 4.10, 4.11).
  - Holographic scanning functor HSP (Definition 4.12).
  - Protocol-stable period data (Definition 3.7).
- **Closed conclusions.**
  - Scan averages realize period values (Corollaries 5.2, 5.4).
  - Auditable error decomposition and truncation bounds (Proposition 6.7, Proposition 6.13).

A.2 Programmatic components (not used as premises)

- **Selection principle.** Conjecture 7.5 proposes a falsifiable stability–complexity functional; it is not used in any proof.
- **Motivic/Langlands compatibility.** Section 8 formulates a commutative-diagram objective factoring functorial upgrades through periods and motives; it is not assumed.

B Additional mathematical notes

B.1 On unique ergodicity of Kronecker rotations

For α satisfying (21), the translation $x \mapsto x + \alpha$ on $\mathbb{T}^d$ is uniquely ergodic with respect to Haar measure. As a consequence, for continuous f, Birkhoff averages converge uniformly in the initial condition. In the main text we used the equivalent uniform distribution formulation because it interfaces directly with discrepancy bounds and Riemann integrable kernels.

B.2 On discrepancy estimates and Diophantine properties

The star discrepancy of the Kronecker sequence $\{x_0 + t\alpha\}$ admits explicit bounds in terms of Diophantine approximation properties of α; in $d = 1$ these are controlled by continued fractions [2,17]. This paper does not require a specific bound, only that discrepancy provides an auditable sampling error term (Corollary 6.3). The selection principle (Section 7) uses these estimates as motivation.

B.3 A closed 1D rate bound for constant-type rotations (bounded partial quotients)

For one-dimensional scans $x_t = x_0 + t\alpha \pmod 1$, finite-horizon stability can be made fully explicit under a standard Diophantine regularity condition.

Proposition B.1 (Star discrepancy for bounded partial quotients)**.** *Let $\alpha = [0; a_1, a_2, \dots]$ be irrational with bounded partial quotients $a_n \le A$. Then the star discrepancy of the Kronecker point set $P_N = \{\{x_0 + t\alpha\} : 0 \le t \le N - 1\} \subset [0, 1)$ satisfies*

$$D_N^*(P_N) \le \frac{2A(3 + \log_\varphi N)}{N}, \tag{60}$$

where $\varphi = (1 + \sqrt{5})/2$.

Proof. Let q_n denote the continued-fraction convergent denominators of α, and write the Ostrowski expansion $N = \sum_{j=0}^{m} b_j q_j$. For $u \in [0,1]$, set $f_u(x) = \mathbf{1}_{[0,u)}(x) - u$, so that $\int_0^1 f_u = 0$ and $\mathrm{Var}(f_u) \le 2$.

By the Denjoy–Koksma inequality at convergent lengths [14, 27, 28], each block sum over a length-q_j orbit segment is bounded by $\mathrm{Var}(f_u)$ uniformly in the starting point. Applying an Ostrowski block decomposition and summing over the b_j blocks yields

$$\left|\sum_{t=0}^{N-1} f_u(x_0 + t\alpha)\right| \le \mathrm{Var}(f_u) \sum_{j=0}^{m} b_j \le 2 \sum_{j=0}^{m} b_j.$$

Dividing by N and taking the supremum over u gives

$$D_N^*(P_N) \le \frac{2}{N} \sum_{j=0}^{m} b_j.$$

If $a_n \le A$, then Ostrowski digits satisfy $b_j \le a_{j+1} \le A$, hence $\sum_{j=0}^{m} b_j \le A(m+1)$. Moreover, $q_{n+1} = a_{n+1} q_n + q_{n-1} \ge q_n + q_{n-1}$ implies $q_n \ge F_n$ for Fibonacci numbers, and $F_n \ge \varphi^{n-2}$ for $n \ge 2$. Since $q_m \le N$, we obtain $\varphi^{m-2} \le N$, hence $m \le 2 + \log_\varphi N$. Therefore $m+1 \le 3 + \log_\varphi N$ and

$$D_N^*(P_N) \le \frac{2A(3 + \log_\varphi N)}{N}.$$

□

Remark B.2. *For the golden branch $\alpha = \varphi^{-1}$ one has $A = 1$, giving an explicit certified $O((\log N)/N)$ discrepancy bound. Combined with Proposition 6.16, this turns the $\log 2$ and π scan realizations into fully quantified, parameter-free finite-N bounds.*

B.4 Hardy–Krause variation (definition)

The Koksma–Hlawka inequality (Theorem 6.1) uses bounded variation in the sense of Hardy–Krause. We recall a standard definition (anchored at 1) and a useful representation for smooth functions.

Definition B.3 (Hardy–Krause variation anchored at 1)**.** *Let $f : [0,1]^d \to \mathbb{R}$. For a nonempty subset $u \subset \{1, \dots, d\}$, write $x_u = (x_j)_{j \in u}$ and let $(x_u; 1)$ denote the point in $[0,1]^d$ obtained by setting coordinates in u to x_u and all remaining coordinates to 1. Define the Hardy–Krause variation anchored at 1 by*

$$\mathrm{Var}_{\mathrm{HK}}(f) := \sum_{\emptyset \ne u \subset \{1,\dots,d\}} V(f(\cdot; 1); u)\,, \tag{61}$$

where $V(f(\cdot;1);u)$ is the Vitali variation of the function $x_u \mapsto f(x_u; 1)$ on $[0,1]^{|u|}$.

Remark B.4 (Smooth-case representation)**.** *If f has continuous mixed partial derivatives of all orders up to d, then one has the standard representation*

$$\mathrm{Var}_{\mathrm{HK}}(f) = \sum_{\emptyset \ne u \subset \{1,\dots,d\}} \int_{[0,1]^{|u|}} \left| \frac{\partial^{|u|}}{\prod_{j \in u} \partial x_j} f(x_u; 1) \right| \mathrm{d}x_u, \tag{62}$$

see, e.g., [3, 4].

B.5 Erdős–Turán–Koksma inequality and a Kronecker discrepancy bound

The following inequality bounds star discrepancy by exponential sums. It provides a closed quantitative bridge from the scan slope α to a finite-N discrepancy certificate.

Theorem B.5 (Erdős–Turán–Koksma inequality (star discrepancy)). *Let $P_N = \{x_0, \ldots, x_{N-1}\} \subset [0,1)^d$ and let $H \geq 1$ be an integer. Define*

$$r(h) := \prod_{j=1}^{d} \max\{1, |h_j|\}, \qquad h = (h_1, \ldots, h_d) \in \mathbb{Z}^d.$$

Then one may take $C_d = (3/2)^d$ such that

$$D_N^*(P_N) \leq C_d \left(\frac{1}{H} + \sum_{\substack{h \in \mathbb{Z}^d \\ 0 < \|h\|_\infty \leq H}} \frac{1}{r(h)} \left| \frac{1}{N} \sum_{t=0}^{N-1} e^{2\pi \mathrm{i} \langle h, x_t \rangle} \right| \right). \tag{63}$$

Definition B.6 (ETK bracket term). *For a point set $P_N = \{x_0, \ldots, x_{N-1}\} \subset [0,1)^d$ and an integer $H \geq 1$, define*

$$B_{N,H}(P_N) := \frac{1}{H} + \sum_{\substack{h \in \mathbb{Z}^d \\ 0 < \|h\|_\infty \leq H}} \frac{1}{r(h)} \left| \frac{1}{N} \sum_{t=0}^{N-1} e^{2\pi \mathrm{i} \langle h, x_t \rangle} \right|. \tag{64}$$

Remark B.7. *With Definition B.6, the inequality* (63) *can be written as*

$$D_N^*(P_N) \leq C_d \, B_{N,H}(P_N), \tag{65}$$

with an explicit admissible choice $C_d = (3/2)^d$.

Remark B.8. *We use Theorem B.5 as a standard tool; the explicit admissible choice $C_d = (3/2)^d$ (and closely related variants) is standard in the quasi-Monte Carlo literature, see, e.g., [3, Ch. 2] or [4, Ch. 3]. In Section 9 we report the explicit bracket term in* (63) *(with a fixed truncation parameter H) and multiply it by this C_d to obtain a fully numerical star-discrepancy certificate.*

Lemma B.9 (Geometric-series bound for Kronecker exponential sums). *Let $\theta \in \mathbb{R}$ and define $\|\theta\| := \min_{m \in \mathbb{Z}} |\theta - m|$ (distance to the nearest integer). Then for every $N \geq 1$,*

$$\left| \sum_{t=0}^{N-1} e^{2\pi \mathrm{i} t\theta} \right| \leq \min\left\{ N, \frac{1}{2\|\theta\|} \right\}. \tag{66}$$

Proof. If $\theta \in \mathbb{Z}$, the left-hand side equals N. Otherwise,

$$\sum_{t=0}^{N-1} e^{2\pi \mathrm{i} t\theta} = \frac{1 - e^{2\pi \mathrm{i} N\theta}}{1 - e^{2\pi \mathrm{i}\theta}},$$

so

$$\left| \sum_{t=0}^{N-1} e^{2\pi \mathrm{i} t\theta} \right| \leq \frac{2}{|1 - e^{2\pi \mathrm{i}\theta}|} = \frac{1}{|\sin(\pi\theta)|}.$$

Using $|\sin(\pi\theta)| \geq 2\|\theta\|$ yields (66). The bound by N is trivial. ☐

Proposition B.10 (Auditable discrepancy bound for Kronecker scans)**.** *Let* $x_t = x_0 + t\alpha \pmod 1$ *with* $\alpha \in \mathbb{R}^d$ *and point set* $P_N = \{x_0, \dots, x_{N-1}\} \subset [0,1)^d$*. Then for every integer* $H \geq 1$*,*

$$D_N^*(P_N) \leq C_d \left(\frac{1}{H} + \sum_{\substack{h \in \mathbb{Z}^d \\ 0 < \|h\|_\infty \leq H}} \frac{1}{r(h)} \min\left\{1, \frac{1}{2N\|\langle h, \alpha\rangle\|}\right\} \right), \tag{67}$$

with C_d *as in Theorem B.5.*

Proof. For $h \in \mathbb{Z}^d$,

$$\sum_{t=0}^{N-1} e^{2\pi \mathrm{i}\langle h, x_t\rangle} = e^{2\pi \mathrm{i}\langle h, x_0\rangle} \sum_{t=0}^{N-1} e^{2\pi \mathrm{i} t\langle h, \alpha\rangle},$$

so the modulus is independent of x_0. Apply Theorem B.5 and bound the exponential sum by Lemma B.9. □

Definition B.11 (Computable Kronecker bracket term)**.** *For* $\alpha \in \mathbb{R}^d$ *and integers* $N, H \geq 1$*, define*

$$B_{N,H}(\alpha; N) := \frac{1}{H} + \sum_{\substack{h \in \mathbb{Z}^d \\ 0 < \|h\|_\infty \leq H}} \frac{1}{r(h)} \min\left\{1, \frac{1}{2N\|\langle h, \alpha\rangle\|}\right\}. \tag{68}$$

Remark B.12 (From ETK to an auditable Kronecker certificate)**.** *For a Kronecker orbit prefix* $P_N = \{x_0 + t\alpha \pmod 1 : 0 \leq t \leq N-1\}$*, Definition B.6 and Lemma B.9 imply*

$$B_{N,H}(P_N) \leq B_{N,H}(\alpha; N),$$

and hence

$$D_N^*(P_N) \leq C_d\, B_{N,H}(\alpha; N). \tag{69}$$

In Section 9 we report the computable quantity $B_{N,H}(\alpha; N)$ *and choose* H *from fixed candidate lists by minimizing it.*

B.6 A Diophantine-rate corollary (explicit)

The computable bracket term $B_{N,H}(\alpha; N)$ is designed for auditability and direct evaluation. In some regimes one may prefer a closed rate bound in terms of Diophantine approximation constants of α. The following corollary is a standard consequence of Proposition B.10 together with a crude but explicit bound on the ETK weight sum.

Definition B.13 (Diophantine condition (sup-norm form))**.** *We say that* $\alpha \in \mathbb{R}^d$ *satisfies a Diophantine condition of type* (c, τ) *if there exist constants* $c > 0$ *and* $\tau \geq 0$ *such that for all* $0 \neq h \in \mathbb{Z}^d$*,*

$$\|\langle h, \alpha\rangle\| \geq \frac{c}{\|h\|_\infty^\tau}. \tag{70}$$

Lemma B.14 (A crude ETK weight-sum bound)**.** *For every integer* $H \geq 1$*,*

$$\sum_{\substack{h \in \mathbb{Z}^d \\ 0 < \|h\|_\infty \leq H}} \frac{1}{r(h)} \leq \left(1 + 2\sum_{k=1}^{H} \frac{1}{k}\right)^d - 1 \leq (3 + 2\log H)^d - 1. \tag{71}$$

Proof. Write the sum over $\|h\|_\infty \leq H$ as a product of one-dimensional sums:

$$\sum_{\substack{h \in \mathbb{Z}^d \\ \|h\|_\infty \leq H}} \frac{1}{r(h)} = \prod_{j=1}^{d} \left(\sum_{m=-H}^{H} \frac{1}{\max\{1, |m|\}} \right) = \left(1 + 2\sum_{k=1}^{H} \frac{1}{k}\right)^d.$$

Removing the $h = 0$ term yields the first inequality. For the second, use $\sum_{k=1}^{H} k^{-1} \le 1 + \log H$ for $H \ge 1$. □

Corollary B.15 (Explicit discrepancy bound under a Diophantine condition)**.** *Assume $\alpha \in \mathbb{R}^d$ satisfies the Diophantine condition* (70) *of type (c, τ) with $\tau > 0$. Let $P_N = \{x_0 + t\alpha \pmod 1 : 0 \le t \le N-1\}$ and let $N \ge 1$. Then for every integer $H \ge 1$,*

$$D_N^*(P_N) \le C_d \left(\frac{1}{H} + \frac{H^\tau}{2Nc} (3 + 2\log H)^d \right), \tag{72}$$

with the same admissible constant $C_d = (3/2)^d$ as in Theorem B.5. In particular, with the explicit choice

$$H := \left\lceil (2Nc)^{\frac{1}{\tau+1}} \right\rceil, \tag{73}$$

one obtains the closed rate bound

$$D_N^*(P_N) \le 2C_d \,(2Nc)^{-\frac{1}{\tau+1}} \,(3 + 2\log H)^d. \tag{74}$$

Proof. By Proposition B.10 and (70),

$$\min\left\{1, \frac{1}{2N\|\langle h, \alpha\rangle\|}\right\} \le \min\left\{1, \frac{\|h\|_\infty^\tau}{2Nc}\right\} \le \frac{H^\tau}{2Nc} \qquad (0 < \|h\|_\infty \le H),$$

which gives (72) after applying Lemma B.14. For (74), the choice (73) implies $H^{-1} \le (2Nc)^{-1/(\tau+1)}$ and $H^\tau/(2Nc) \le (2Nc)^{-1/(\tau+1)}$. Substitute these into (72) and absorb constants. □

Remark B.16. *Sharper bounds are available in the discrepancy literature by using refined Diophantine information and sharper ETK variants; see, e.g., [2–4, 6]. The purpose of Corollary B.15 is to provide a fully explicit closed certificate within the present audit style.*

C Reproducibility notes

The scripts used in Section 9 are included in the paper directory:

- `scripts/period_scan_experiments.py` generates scan-period estimates for $\log 2$, π, $\zeta(2)$ and $\zeta(3)$ and writes LaTeX table rows into `sections/generated/`. In particular it writes `log2_rows.tex`, `pi_rows.tex`, `zeta2_rows.tex`, `zeta3_rows.tex` and the auxiliary fit snippets `log2_fit.tex`, `pi_fit.tex`.

- `scripts/alpha_inverse_search.py` performs an exhaustive bounded-complexity enumeration in the ansatz $a\pi^3 + b\pi^2 + c\pi$ and writes the top candidates into `sections/generated/alpha_integer_search_rows.tex`.

All randomness is controlled by fixed seeds. The numerical values in the main text tables are obtained by executing these scripts on a standard Python 3 interpreter. For the multi-dimensional discrepancy certificates, the scripts compute the explicit Kronecker bracket term $B_{N,H}(\alpha; N)$ (Definition B.11), choose H from fixed candidate lists by minimizing it, and use the explicit ETK constant $C_d = (3/2)^d$ (Theorem B.5) to report fully numerical star-discrepancy and sampling certificates.

D Worked motivic examples: periods with Frobenius-trace data

Section 8 formulates a compatibility target in which the closed functor HSP : $\mathsf{Scan}_{\mathrm{alg}} \to \mathsf{PerDatum}$ is followed by (programmatic) motive and realization arrows. The goal of this appendix is to anchor that target by recording two classical, fully worked test cases in which:

- a period datum is canonically associated to a standard geometric source;
- the same source carries ℓ-adic realizations with Frobenius traces (hence an Euler product and an L-function);
- the corresponding automorphic object is explicit.

Nothing in this appendix is used in the closed proof chain of the paper; it serves only as a controlled reference point for what the arrows in (58) mean in standard arithmetic geometry.

D.1 Mixed Tate test case: $\mathbb{P}^1 \setminus \{0, 1, \infty\}$ and iterated integrals

Let $X := \mathbb{P}^1 \setminus \{0, 1, \infty\}$. A standard period calculus on X is generated by the logarithmic 1-forms

$$\omega_0 := \frac{\mathrm{d}t}{t}, \qquad \omega_1 := \frac{\mathrm{d}t}{1-t},$$

and their iterated integrals along paths between rational points. These iterated integrals are periods of the (unipotent) fundamental group of X and lie in the period algebra of mixed Tate motives over $\mathbb{Z}$ (and more generally over $\mathbb{Z}[1/N]$ when endpoints involve N) [9, 10, 29].

A rational-endpoint period for $\log 2$. The basic scan-period example $\int_0^1 (1+x)^{-1} \mathrm{d}x = \log 2$ can be rewritten as an iterated-integral period on X with rational endpoints. Indeed, with the change of variables $x = t/(1-t)$ one has $\mathrm{d}x/(1+x) = \mathrm{d}t/(1-t)$ and

$$\log 2 = \int_0^1 \frac{\mathrm{d}x}{1+x} = \int_0^{1/2} \frac{\mathrm{d}t}{1-t} = \int_0^{1/2} \omega_1. \tag{75}$$

Thus $\log 2$ is realized as a period datum (D, f, Id) with $D = [0, 1/2] \subset \mathbb{R}$ and $f(t) = 1/(1-t) \in \mathbb{Q}(t)$, which lies in the controlled cubical setting of this paper.

Multiple zeta values as periods on X. Multiple zeta values admit a well-known iterated integral representation on X (see, e.g., [9, 10]): for integers $n \geq 2$,

$$\zeta(n) = \int_0^1 \omega_1\, \omega_0^{n-1}. \tag{76}$$

In particular,

$$\zeta(2) = \int_0^1 \omega_1\, \omega_0 = \int_{0<t_1<t_2<1} \frac{\mathrm{d}t_1}{1-t_1} \frac{\mathrm{d}t_2}{t_2}, \tag{77}$$

which is a Kontsevich–Zagier period with a $\mathbb{Q}$-semialgebraic domain (a simplex) and a rational integrand.

Interpretation for the compatibility target. In this test case, the "motive" arrow in (58) can be instantiated within the standard mixed Tate framework: the motivic source is a mixed Tate object attached to X (e.g. via the unipotent fundamental group), and its periods include the iterated integrals above. The point of recording (75)–(77) is that they provide a concrete bridge from our cubical period data to a motivic organization where period relations are governed by a well-studied structure [10, 29].

D.2 Modular/elliptic test case: weight-2 newforms, periods, and Frobenius traces

Let $f(z) = \sum_{n\geq 1} a_n\, q^n$ be a normalized weight-2 newform on $\Gamma_0(N)$, with $q = e^{2\pi \mathrm{i} z}$. Its L-series is

$$L(f,s) := \sum_{n=1}^{\infty} \frac{a_n}{n^s}, \qquad \Re(s) \gg 1.$$

The Mellin transform gives the classical period integral expression (see, e.g., [24, 30])

$$(2\pi)^{-s}\Gamma(s)\, L(f,s) = \int_0^{\infty} f(\mathrm{i}t)\, t^{s-1}\, \mathrm{d}t, \tag{78}$$

and in particular at $s = 1$,

$$L(f,1) = 2\pi \int_0^{\infty} f(\mathrm{i}t)\, \mathrm{d}t. \tag{79}$$

Equivalently, the holomorphic differential $\omega_f := 2\pi \mathrm{i} f(z)\, \mathrm{d}z$ on the modular curve $X_0(N)$ has periods given by integrals of ω_f along 1-cycles (modular symbols), and these periods control special values such as $L(f,1)$.

Frobenius traces from ℓ-adic realizations. Deligne attaches to f an ℓ-adic Galois representation $\rho_{f,\ell}$ whose Frobenius traces recover the Hecke eigenvalues [22, 23]. Concretely, for primes $p \nmid N\ell$,

$$\mathrm{Tr}\big(\rho_{f,\ell}(\mathrm{Frob}_p)\big) = a_p, \tag{80}$$

and the Euler factors of $L(f,s)$ are determined by these traces.

Elliptic curve specialization. When the Hecke field of f is $\mathbb{Q}$, the modular abelian variety attached to f is an elliptic curve $E/\mathbb{Q}$ and the motive may be identified with $H^1(E)$; in this case (80) specializes to the familiar point-counting identity

$$a_p = p + 1 - \#E(\mathbb{F}_p),$$

and the period integrals of the Néron differential on E yield the real/imaginary periods of E. The equality $L(E,s) = L(f,s)$ provides a concrete instance of the "period–motive–Frobenius" chain in (58).

Interpretation for the compatibility target. This modular test case is a regime in which the arrows "period data $\to$ motive $\to$ ℓ-adic realization" and "motive $\to$ automorphic representation" are classical. It therefore provides a canonical benchmark for any future attempt to connect scan-realized period data (via HSP) with Hecke/Frobenius trace data extracted from protocol-level constructions.

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