Holographic Polar Omega Theory: An Axiomatic Upgrade and the Continuous–Discrete Bridge

Haobo Ma*

January 1, 2026

Abstract

We give a short axiomatic upgrade of Omega Theory that makes the continuous–discrete connection explicit and operational. The bridge is made concrete via intrinsic scan dynamics (unitary scanning), finite-resolution projection readout via window kernels, and a canonical integer-time coding (Ostrowski numeration, specializing to Zeckendorf in the golden case). The upgrade isolates two structural sources of "quantumness" in the regulated description: (i) intrinsic noncommutativity from a Weyl pair tied to the scan, and (ii) probability measures induced by finite-resolution readout (instrument and POVM structure), rather than external sampling postulates. We also fix a canonical regularization convention for regulated-to-continuum passages via orbit traces and Abel finite parts. Detailed mathematical constructions appear in the companion tool-paper [1], while the full physics manuscript develops the micro-ontology, phenomenology, and cosmological templates [2].

Keywords: Scan–projection readout, Weyl pair, Ostrowski numeration, Zeckendorf decomposition, Sturmian sequences, cut-and-project quasicrystals, induced measures, finite-part regularization.

Contents

*Email: auric@aelf.io

1 Positioning and scope

This note isolates the minimal axiomatic upgrades needed to make the continuous–discrete bridge explicit in Omega Theory. The guiding viewpoint is that a finite observer does not access an external time parameter, nor an infinitely refined continuum: operational time and statistics arise from a scan–projection readout of intrinsic phase data at finite resolution.

We treat the scan algebra and its readout as internal, not auxiliary: by holographic encoding (O4) one may realize the observer-accessible sector and pointer/readout observables within boundary degrees of freedom on the relevant code subspace, and by causal locality (O3) their tick iteration is an intrinsic automorphism rather than an externally supplied clock parameter.

Division of labor. The companion physics manuscript [2] develops the full regulated micro-ontology (QCA on quasicrystal substrates, holographic encoding, cosmological templates, and interpretational structure). The companion mathematical tool-paper [1] develops self-contained constructions behind the scan operator, window projection readout, Ostrowski/Zeckendorf coordinates, and orbit-calculus/finite-part regularization. The present note only records the upgraded axioms and the shortest consequence chain that connects them.

What is new here. Relative to the baseline axioms (static universe state, finite information, causal locality, and holographic encoding), we add explicit axioms for scan–projection readout, intrinsic noncommutativity (Weyl pair), induced probability measures from readout (instrument/POVM structure), and a canonical orbit-regularization convention.

2 Upgraded axioms

We keep the baseline axioms O1–O4 and record the upgrade axioms O5–O6, together with a regularization convention.

Axiom 2.1 (O1: Omega axiom)**.** *The physical universe is specified by a unique normalized global state ω_Ω on a quasi-local operator algebra $\mathfrak{A}$ associated with a countable graph of finite-dimensional degrees of freedom. There is no external time parameter; the fundamental description is a single state rather than a family $\{\omega(t)\}$. Any "dynamics" is encoded as intrinsic automorphisms acting on observables (O3), while operational clock-time is specified only after a scan–projection readout is chosen (O5).*

Axiom 2.2 (O2: Finite information)**.** *For any causally closed region with boundary area A, the effective dimension of the regulated state space obeys a holographic bound of the form*

$$\dim \mathcal{H}_{\text{region}} \leq \exp(A/4\ell_P^2). \tag{1}$$

Axiom 2.3 (O3: Causal locality)**.** *There exists a discrete-step causal update given by a unital $*$-automorphism $\mathcal{U} : \mathfrak{A} \to \mathfrak{A}$. In the regulated representation it is implemented by a unitary, $\mathcal{U}(A) = U^*AU$, and has finite-range causal propagation: local observables evolve to observables supported only on bounded neighborhoods. Correlators are thus of Heisenberg/relational form $\omega_\Omega(\mathcal{U}^n(A))$ for $n \in \mathbb{Z}$, without introducing a family of time-labeled states.*

Axiom 2.4 (O4: Holographic mapping)**.** *There exists a holographic encoding map Φ from bulk to boundary that is (approximately) isometric on the relevant code subspace and supports approximate operator-algebra quantum error correction / entanglement-wedge reconstruction. In particular, observer-accessible readout operators may be chosen within the boundary algebra on the code subspace, and the effective observer state ω_{eff} in O5 is obtained by restricting ω_Ω (after encoding) to those observables.*

Axiom 2.5 (O5: Scan–projection readout (and induced measure)). *Any operational notion of time available to a finite observer is obtained from a finite-resolution scan and projection of intrinsic phase data. In particular, there exists a circle-valued phase readout $x(\tau) \in \mathbb{R}/\mathbb{Z}$ of an intrinsic scan parameter τ and a stroboscopic sampling at ticks $t \in \mathbb{Z}_{\geq 0}$ (counting scan steps) such that the sampled readout $x_t := x(\tau_t)$ is modeled (in the generic aperiodic regime) by an irrational circle rotation with slope $\alpha \in (0,1) \setminus \mathbb{Q}$:*

$$x_t = x_0 + t\alpha \pmod 1. \tag{2}$$

In the continuum covariant model of O6, x is the spectral coordinate of the pointer mode V and the scan acts as the translation $x \mapsto x + \alpha$, so the rotation model can be viewed as the induced dynamics on the pointer spectrum.

In regulated finite settings, one may work with long-period rational approximants $\alpha_n = p_n/q_n$ (e.g. convergents of the continued fraction of α), so that the irrational rotation describes the scaling/continuum limit $q_n \to \infty$ and the effective aperiodic regime at finite resolution.

Moreover, finite-resolution readout induces probabilities as a projection measure rather than as an external sampling postulate: for each resolution parameter $\varepsilon > 0$ there is a family of effects $\{E_k^{(\varepsilon)}\}$ with $\sum_k E_k^{(\varepsilon)} = I$ and outcome statistics are determined by

$$P_k^{(\varepsilon)} = \omega_{\mathrm{eff}}\left(E_k^{(\varepsilon)}\right). \tag{3}$$

In a Hilbert-space representation, $\omega_{\mathrm{eff}}(A) = \mathrm{Tr}(\rho A)$ for a density matrix ρ on $\mathcal{H}_{\mathrm{eff}}$. A convenient model for finite-resolution readout is a unitary pointer mode V with spectral measure Π_V on $\mathbb{R}/\mathbb{Z}$ (so that $V = \int_{\mathbb{R}/\mathbb{Z}} \mathrm{e}^{2\pi \mathrm{i} x} \, d\Pi_V(x)$); equivalently $V = \mathrm{e}^{2\pi \mathrm{i} X}$ for a circle-valued pointer observable X. Given response functions $w_k^{(\varepsilon)} : \mathbb{R}/\mathbb{Z} \to [0,1]$ with $\sum_k w_k^{(\varepsilon)}(x) = 1$, the induced effects are

$$E_k^{(\varepsilon)} = \int_{\mathbb{R}/\mathbb{Z}} w_k^{(\varepsilon)}(x) \, d\Pi_V(x). \tag{4}$$

To describe sequential readouts, one may fix an associated instrument $\{\mathcal{I}_k^{(\varepsilon)}\}$ (completely positive maps) such that $\sum_k \mathcal{I}_k^{(\varepsilon)}$ is trace preserving and $\mathrm{Tr}(\mathcal{I}_k^{(\varepsilon)}(\rho)) = \mathrm{Tr}(\rho E_k^{(\varepsilon)})$. In finite dimensions an operator-sum form exists,

$$\mathcal{I}_k^{(\varepsilon)}(\rho) = \sum_a K_{k,a}^{(\varepsilon)} \, \rho \, K_{k,a}^{(\varepsilon)*}, \qquad E_k^{(\varepsilon)} = \sum_a K_{k,a}^{(\varepsilon)*} K_{k,a}^{(\varepsilon)}. \tag{5}$$

In the sharp limit $w_k^{(\varepsilon)} \to \mathbb{1}_{W_k}$ the effects reduce to projectors (window projections) and conditioning reduces to the standard Lüders/Kraus instrument.

Axiom 2.6 (O6: Unitary scan algebra (Weyl pair)). *The scan underlying tick-labeled readouts is implemented by a nontrivial unitary U_{scan} on an effective observer sector $\mathcal{H}_{\mathrm{eff}}$. The scan is compatible with the causal update (O3) on the observer-accessible algebra: there exists an observer-accessible subalgebra $\mathfrak{A}_{\mathrm{eff}} \subseteq \mathfrak{A}$ (or in the boundary algebra under Φ) and a representation $\pi_{\mathrm{eff}} : \mathfrak{A}_{\mathrm{eff}} \to \mathcal{B}(\mathcal{H}_{\mathrm{eff}})$ such that*

$$\pi_{\mathrm{eff}}(\mathcal{U}(A)) = U_{\mathrm{scan}}^* \, \pi_{\mathrm{eff}}(A) \, U_{\mathrm{scan}}, \qquad A \in \mathfrak{A}_{\mathrm{eff}}. \tag{6}$$

Moreover, there exists a conjugate unitary V (a phase/pointer mode) such that

$$U_{\mathrm{scan}} V = \mathrm{e}^{2\pi \mathrm{i} \alpha} V U_{\mathrm{scan}}, \tag{7}$$

with the same irrational slope α as in O5. Consequently there are no nonzero states that are simultaneous eigenvectors of U_{scan} and V, and quantitative uncertainty tradeoffs follow from the Weyl relation.

Equivalently, $U_{\mathrm{scan}}^{t} V U_{\mathrm{scan}}^{-t} = \mathrm{e}^{2\pi \mathrm{i} t\alpha} V$*. If* $V = \mathrm{e}^{2\pi \mathrm{i} X}$ *for a circle-valued pointer observable* X*, then* $X \mapsto X + \alpha \pmod 1$ *under the scan, matching the rotation model in O5.*

Continuum model. *A canonical covariant realization of the Weyl relation is on* $L^2(\mathbb{R}/\mathbb{Z})$ *with*

$$(U_{\mathrm{scan}}\psi)(x) = \psi(x+\alpha), \qquad (V\psi)(x) = \mathrm{e}^{2\pi \mathrm{i} x}\psi(x), \tag{8}$$

so that $U_{\mathrm{scan}} V = \mathrm{e}^{2\pi \mathrm{i}\alpha} V U_{\mathrm{scan}}$ *and the induced action on the spectrum is the circle rotation* $x \mapsto x + \alpha$*.*

Regulated realizations. *In a strictly finite-dimensional regulated observer sector of dimension* d*, an exact Weyl relation* $U_{\mathrm{scan}} V = \mathrm{e}^{2\pi \mathrm{i}\alpha} V U_{\mathrm{scan}}$ *forces* $\mathrm{e}^{2\pi \mathrm{i}\alpha d} = 1$ *(e.g. by taking determinants), hence* $\alpha \in \mathbb{Q}$*. The irrational slope* α *is therefore to be understood as a limiting/continuum parameter: regulated realizations use rational convergents* $\alpha_n = p_n/q_n$ *with exact* q_n*-dimensional clock/shift Weyl pairs, and the irrational rotation algebra is recovered as* $q_n \to \infty$ *(see [1]).*

Convention R1 (Orbit regularization and finite parts). Effective continuum observables are defined as limits of regulated scan-orbit functionals: orbit averages are taken as orbit traces (ergodic/Haar averages on orbit closures), while divergent orbit sums are assigned a canonical finite part via Abel regularization and pole subtraction [1]. When multiple regulators are present (finite dimension q, finite resolution ε, Abel parameter r), the finite-part assignment is defined at fixed (q, ε) by the Abel limit $r \uparrow 1$ and pole subtraction, and only afterwards one takes scaling/continuum limits. In this note we treat $r \uparrow 1$ as the innermost limit, take $q \to \infty$ along convergents when passing from rational to irrational slope, and take the sharp readout limit $\varepsilon \downarrow 0$ only when a projective/window idealization is required. When different paths in (q, ε) lead to different limits, this convention fixes the canonical path/order.

Assumption 2.7 (R2: Existence of canonical scaling limits)**.** *For the class of scan-orbit functionals and readout kernels considered, the iterated limits specified in Convention R1 exist along the canonical path/order, and define the effective continuum quantities used below.*

3 Minimal consequence chain

We record the shortest implication chain that ties the new axioms together; detailed constructions and proofs are given in [1] and in the companion physics manuscript [2].

3.1 Relational ticks from scan covariance

By O3, correlators are defined by iterating the intrinsic update $\mathcal{U}$ on observables under the single state ω_Ω. By O6, on the observer-accessible algebra $\mathfrak{A}_{\mathrm{eff}}$ the same iteration is implemented by the scan unitary U_{scan} in the effective representation. Hence tick-labeled readouts are naturally modeled as orbits under U_{scan}: for any finite-resolution effect $E_k^{(\varepsilon)}$,

$$P_k^{(\varepsilon)}(t) = \omega_{\mathrm{eff}}\left(U_{\mathrm{scan}}^{*t} E_k^{(\varepsilon)} U_{\mathrm{scan}}^{t}\right), \tag{9}$$

and, for a circle pointer mode $V = \mathrm{e}^{2\pi \mathrm{i} X}$, the Weyl relation implies $X \mapsto X + \alpha \pmod 1$ under each tick.

The formula above gives the *unconditioned* single-tick statistics in the static state. For sequential readout (conditioning/back-action), one uses the instrument structure from O5, interleaved with the scan evolution.

3.2 From scan–projection to canonical coding

Axiom O5 gives a canonical map from a continuous intrinsic parameter to a discrete symbolic stream once a readout partition and resolution are fixed. A convenient two-outcome specialization uses the sharp windows $W_L = [0, 1 - \alpha)$ and $W_S = [1 - \alpha, 1)$ (exchanging $L \leftrightarrow S$ corresponds to the complementary partition). At finite resolution ε, choose response functions $w_L^{(\varepsilon)}, w_S^{(\varepsilon)} : \mathbb{R}/\mathbb{Z} \to [0, 1]$ with $w_L^{(\varepsilon)} + w_S^{(\varepsilon)} = 1$ and $w_{L,S}^{(\varepsilon)} \to \mathbb{1}_{W_{L,S}}$ as $\varepsilon \downarrow 0$, and let $E_{L,S}^{(\varepsilon)}$ be the induced effects as in O5. The tick-t symbol distribution is then

$$\mathbb{P}_\varepsilon(\sigma_t = S) = \omega_{\text{eff}}\left(U_{\text{scan}}^{*t} E_S^{(\varepsilon)} U_{\text{scan}}^t\right), \qquad \mathbb{P}_\varepsilon(\sigma_t = L) = \omega_{\text{eff}}\left(U_{\text{scan}}^{*t} E_L^{(\varepsilon)} U_{\text{scan}}^t\right). \tag{10}$$

More generally, the associated instrument determines the joint law of a finite symbol string by sequential composition; the above are the single-tick marginals. Concretely, let $\mathcal{U}_{\text{scan}}(\rho) := U_{\text{scan}} \rho U_{\text{scan}}^*$. With the convention that a scan step separates successive readouts, the joint probability of a length-T outcome string $(k_0, \ldots, k_{T-1})$ is

$$\mathbb{P}_\varepsilon(k_0, \ldots, k_{T-1}) = \operatorname{Tr}\left(\mathcal{I}_{k_{T-1}}^{(\varepsilon)} \circ \mathcal{U}_{\text{scan}} \circ \cdots \circ \mathcal{I}_{k_0}^{(\varepsilon)}(\rho)\right), \tag{11}$$

where ρ represents ω_{eff} on $\mathcal{H}_{\text{eff}}$. In a commutative pointer model where X has a sharp value x_t at tick t, the marginals reduce to $\mathbb{P}_\varepsilon(\sigma_t = S) = w_S^{(\varepsilon)}(x_t)$ and $\mathbb{P}_\varepsilon(\sigma_t = L) = w_L^{(\varepsilon)}(x_t)$.

In the sharp (window) limit $w_{L,S}^{(\varepsilon)} \to \mathbb{1}_{W_{L,S}}$ the effects become window projectors. In the commutative/classical pointer idealization where the pointer phase follows the rotation model $x_t = x_0 + t\alpha \pmod 1$, this reduces to the deterministic window coding: define the binary word $\sigma_t \in \{L, S\}$ by

$$\sigma_t := \begin{cases} L, & x_t \in W_L, \\ S, & x_t \in W_S. \end{cases} \tag{12}$$

Equivalently, letting $s_t := \lfloor x_0 + (t+1)\alpha \rfloor - \lfloor x_0 + t\alpha \rfloor \in \{0, 1\}$, one may take $\sigma_t = S$ iff $s_t = 1$ and $\sigma_t = L$ iff $s_t = 0$ (mechanical word form). For irrational α this produces a Sturmian word of slope α; changing x_0 shifts the word (and may affect only finitely many symbols when x_0 lies on a window boundary). A canonical representative is the characteristic word obtained by the choice $x_0 = 0$. The same continued-fraction data of $\alpha = [0; a_1, a_2, \ldots]$ yields a canonical integer coordinate system for tick indices (Ostrowski numeration): writing p_n/q_n for the convergents, every $t \in \mathbb{Z}_{\geq 0}$ has a unique expansion

$$t = \sum_{n=0}^{N} b_n q_n, \qquad 0 \leq b_n \leq a_{n+1}, \qquad b_n = a_{n+1} \Rightarrow b_{n-1} = 0. \tag{13}$$

Proposition 3.1 (Golden specialization). *For the golden slope choice $\alpha = \phi^{-2}$ (with $\phi = (1 + \sqrt{5})/2$), the induced Sturmian coding is the Fibonacci word and the Ostrowski numeration specializes to Zeckendorf coding (no adjacent Fibonacci summands).*

Proof sketch. For $x_0 = 0$ the coding above is the (characteristic) mechanical word of slope α. When $\alpha = \phi^{-2} = [0; 2, 1, 1, 1, \ldots]$ this mechanical word is the Fibonacci word. The corresponding convergent denominators satisfy $q_{n+1} = q_n + q_{n-1}$, hence are Fibonacci numbers (up to index shift), and the Ostrowski admissibility constraint reduces to digits $b_n \in \{0, 1\}$ with no adjacent 1's, i.e. Zeckendorf decomposition. □

3.3 Unitary scanning, Weyl pairs, and intrinsic uncertainty

Axiom O6 postulates a Weyl pair (U_{scan}, V) with irrational commutation phase. This is the minimal algebraic source of incompatibility between scan-time and phase/pointer localization.

Proposition 3.2 (No common eigenvectors)**.** *If* $U_{\mathrm{scan}}V = \mathrm{e}^{2\pi\mathrm{i}\alpha}VU_{\mathrm{scan}}$ *with* $\alpha \notin \mathbb{Q}$*, then* U_{scan} *and* V *have no nonzero common eigenvector.*

Proof. Suppose $\psi \neq 0$ is a common eigenvector, $U_{\mathrm{scan}}\psi = u\psi$ and $V\psi = v\psi$ with $|u| = |v| = 1$. Then $U_{\mathrm{scan}}V\psi = uv\psi$ while the Weyl relation gives $U_{\mathrm{scan}}V\psi = \mathrm{e}^{2\pi\mathrm{i}\alpha}VU_{\mathrm{scan}}\psi = \mathrm{e}^{2\pi\mathrm{i}\alpha}vu\psi$. Thus $uv = \mathrm{e}^{2\pi\mathrm{i}\alpha}uv$, forcing $\mathrm{e}^{2\pi\mathrm{i}\alpha} = 1$, contradicting $\alpha \notin \mathbb{Q}$. □

3.4 Finite-dimensional rational approximants

In regulated finite-dimensional realizations compatible with the holographic cutoff (O2), one works with rational slopes $\alpha = p/q$. Exact Weyl pairs exist in dimension q, and irrational slopes are recovered as scaling limits along convergents $\alpha_n = p_n/q_n$; notably, the same denominators q_n also underlie the Ostrowski coordinates of tick indices. Operationally, O2 bounds the available effective dimension in any finite observer region, and hence bounds the accessible denominators q of exact Weyl pairs; the irrational α is therefore observed only through long-period rational approximants and finite prefixes of the limiting symbolic stream.

Proposition 3.3 (Rational Weyl pair (clock/shift))**.** *Let* $\alpha = p/q$ *with* $\gcd(p, q) = 1$*. On* $\mathbb{C}^q$ *define unitaries* V *and* U *by*

$$(V\psi)_j = \mathrm{e}^{2\pi\mathrm{i}j/q}\psi_j, \qquad (U\psi)_j = \psi_{j+1 \pmod q}. \tag{14}$$

Then $U^pV = \mathrm{e}^{2\pi\mathrm{i}p/q}VU^p$*.*

Proof. A direct computation gives $(UV\psi)_j = \mathrm{e}^{2\pi\mathrm{i}(j+1)/q}\psi_{j+1}$ and $(VU\psi)_j = \mathrm{e}^{2\pi\mathrm{i}j/q}\psi_{j+1}$, hence $UV = \mathrm{e}^{2\pi\mathrm{i}/q}VU$ and therefore $U^pV = \mathrm{e}^{2\pi\mathrm{i}p/q}VU^p$. □

On the symbolic side, the same rational approximants yield periodic mechanical words: for $\alpha = p/q$ and window coding as above, $x_{t+q} = x_t$ and hence $\sigma_{t+q} = \sigma_t$. Over one period one has

$$\#\{0 \leq t < q : \sigma_t = S\} = \sum_{t=0}^{q-1}\left(\lfloor (t+1)\alpha \rfloor - \lfloor t\alpha \rfloor\right) = \lfloor q\alpha \rfloor = p, \tag{15}$$

so the period block contains p symbols S and $q - p$ symbols L (a Christoffel/mechanical rational block). Along convergents $\alpha_n = p_n/q_n \to \alpha$ these periodic blocks converge in the product topology to the characteristic Sturmian word of slope α (see [1]).

3.5 Rotation algebra viewpoint and canonical trace

The Weyl relation in O6 defines the (irrational) rotation algebra $\mathcal{A}_\alpha$, the universal C^*-algebra generated by unitaries U, V with $UV = \mathrm{e}^{2\pi\mathrm{i}\alpha}VU$ (a noncommutative 2-torus). For irrational α, $\mathcal{A}_\alpha$ admits a canonical tracial state τ characterized on Fourier monomials by

$$\tau(U^mV^n) = \begin{cases} 1, & (m, n) = (0, 0), \\ 0, & \text{otherwise,} \end{cases} \tag{16}$$

and this trace is unique among tracial states [3, 4]. In the covariant continuum model of O6, restricting τ to the commutative subalgebra generated by V recovers the Haar average on the circle, matching the orbit-trace normalization used in Convention R1.

For rational $\alpha = p/q$, the corresponding rotation algebra is represented by the q-dimensional clock/shift Weyl pair (hence by a matrix algebra), and the canonical trace reduces to the normalized matrix trace, matching the regulated averaging prescription.

3.6 One mechanism for both time readout and quasicrystal texture

The scan–projection mechanism is not restricted to "time." Replacing the tick index by a spatial index along a lattice geodesic yields Sturmian/Fibonacci textures, while interpreting the same window data as an acceptance window yields a cut-and-project (model-set) geometry. In this way, Fibonacci coin textures in a 1D reduction, phason shifts, and Zeckendorf-clocked drives can be traced to the same scan–projection source (see [2] for the QCA setting).

3.7 Projection kernels and induced measures (Born/instrument structure)

Axiom O5 makes probability a derived object of finite-resolution readout: once the effects $\{E_k^{(\varepsilon)}\}$ are fixed by the readout kernel, the state functional ω_{eff} induces the outcome law $P_k^{(\varepsilon)} = \omega_{\text{eff}}(E_k^{(\varepsilon)})$. In regulated finite-dimensional realizations, representing ω_{eff} by a density matrix immediately yields the Born form $P_k^{(\varepsilon)} = \mathrm{Tr}(\rho E_k^{(\varepsilon)})$.

Conversely, if one starts from an abstract assignment of probabilities to (generalized) measurement effects and imposes standard consistency requirements (normalization, additivity under coarse-graining, and noncontextuality), Gleason-type theorems for POVMs (Busch) constrain the assignment to be of trace form [5, 6].

3.8 Orbit traces and Abel finite parts

Convention R1 fixes a canonical normalization for regulated-to-continuum passages: orbit averages become Haar/orbit traces, and constant terms in divergent orbit sums are fixed by Abel finite parts.

Proposition 3.4 (Orbit trace for irrational rotations)**.** *Let $\alpha \notin \mathbb{Q}$ and $x_t = x_0 + t\alpha \pmod 1$. Then the rotation is uniquely ergodic with invariant measure dx, and for any continuous f one has*

$$\lim_{N\to\infty} \frac{1}{N} \sum_{t=0}^{N-1} f(x_t) = \int_0^1 f(x)\, dx. \tag{17}$$

Proof sketch. It suffices to check the claim on Fourier modes $f(x) = \mathrm{e}^{2\pi i m x}$ and then use density of trigonometric polynomials. For $m = 0$ the statement is trivial; for $m \neq 0$ the finite average is a geometric sum and its magnitude is $O(1/N)$ because $\mathrm{e}^{2\pi i m\alpha} \neq 1$ for irrational α. □

Accordingly, in the irrational rotation regime the orbit trace appearing in Convention R1 is the Haar average $\int_0^1 f(x)\, dx$, independent of x_0. For Abel regularization, define

$$S(r) := \sum_{t\geq 0} r^t f(x_t), \qquad 0 < r < 1, \tag{18}$$

and note that the Abel-averaged orbit mean is $(1-r)S(r)$; for continuous f one has $(1-r)S(r) \to \int_0^1 f(x)\, dx$ as $r \uparrow 1$. The finite part prescription fixes the constant term after subtracting the pole:

$$\mathrm{FP} \sum_{t\geq 0} f(x_t) \;:=\; \lim_{r\uparrow 1} \left(S(r) - \frac{1}{1-r} \int_0^1 f(x)\, dx \right), \tag{19}$$

whenever the limit exists; existence along the canonical scaling path/order is part of Assumption 2.7.

Example. For $f \equiv 1$, $S(r) = (1-r)^{-1}$ and $\mathrm{FP} \sum_{t\geq 0} 1 = 0$. For the Fourier mode $f(x) = \mathrm{e}^{2\pi i m x}$ with $m \neq 0$, the orbit average vanishes and

$$S(r) = \frac{\mathrm{e}^{2\pi i m x_0}}{1 - r\, \mathrm{e}^{2\pi i m\alpha}} \xrightarrow[r\uparrow 1]{} \frac{\mathrm{e}^{2\pi i m x_0}}{1 - \mathrm{e}^{2\pi i m\alpha}}, \tag{20}$$

so the Abel finite part coincides with the Abel limit.

4 Discussion and pointers

Why this upgrade matters. The upgrade axioms O5–O6 (together with Convention R1) remove several recurring ambiguities in static-universe frameworks: (i) they make "time" an explicit scan–projection readout rather than an external label; (ii) they identify an intrinsic algebraic origin of incompatibility/uncertainty via a Weyl pair; (iii) they treat probability as induced by readout kernels/instruments; and (iv) they fix a regularization convention tied to the scan orbit.

In the holographic setting (O4), the same upgrade also clarifies where the operational structures live: the effective scan/readout sector can be taken as boundary-local on the code subspace, so that the resulting discrete symbolic streams and induced measures are compatible with entanglement-wedge reconstruction and with the regulated holographic description.

Where to look for details. For mathematical constructions (scan operator realizations, window projections, Ostrowski/Zeckendorf coordinates, and orbit-calculus/finite-part tools), see [1]. For the physics program (QCA on quasicrystals, holographic encoding, continuum limits, and phenomenological templates), see [2].

References

[1] Haobo Ma. Holographic polar arithmetic: Multiplicative ontology, unitary scanning, and the geometric origin of quantum uncertainty. Companion mathematical manuscript, 2025.

[2] Haobo Ma. Omega theory: Axiomatic foundations of holographic spacetime and interactive evolution. Companion physics manuscript, 2025.

[3] Marc A. Rieffel. C*-algebras associated with irrational rotations. *Pacific Journal of Mathematics*, 93(2):415–429, 1981.

[4] Alain Connes. *Noncommutative Geometry*. Academic Press, 1994.

[5] Andrew M. Gleason. Measures on the closed subspaces of a hilbert space. *Journal of Mathematics and Mechanics*, 6(6):885–893, 1957.

[6] Paul Busch. Quantum states and generalized observables: A simple proof of gleason's theorem. *Physical Review Letters*, 91(12):120403, 2003.